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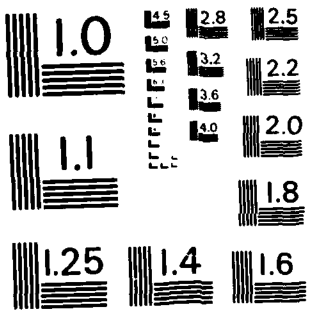

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DISTRIBUTED DETECTION OF SIGNAL WAVEFORMS IN ADDITIVE GAUSSIAN OBSERVATION NOISE*

by

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Dr. N.R. Sandell, Jr.‡

ABSTRACT

This paper is concerned with the detection of signal waveforms by a distributed surveillance network comprised of: (1) a collection of spatially separated sensors, and (2) local signal processors collocated with the sensors. The local signal processors are assumed to implement likelihood ratio tests to detect the presence or absence of the signals. Signal detections may be used for local decisionmaking or passed upward to a fusion center for further processing. In either case, the local detection thresholds cannot be determined independently, but must be determined jointly to optimize overall surveillance system performance. Results are presented concerning the nature of this threshold computation for a number of interesting cases.

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This paper is concerned with the detection of signal waveforms by a distributed surveillance network comprised of: (1) a collection of spatially separated sensors, and (2) local signal processors collocated with the sensors. The local signal processors are assumed to implement likelihood ratio tests to detect the presence of the signals. Signal detections may be used for local decisionmaking or passed upward to a fusion center for further processing. In either case, the local detection thresholds cannot be determined independently but must be determined jointly to optimize overall surveillance system (over)

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SECTION 1

INTRODUCTION

Classical detection theory (see, e.g., [1]-[4]*) has been motivated primarily by single sensor detection problems. Although the signal processing solutions of classical detection theory are in principle equally applicable to multiple sensor detection problems, in practice these solutions may require the communication of raw received signals from physically remote sensors to a central processing location. In many surveillance systems, particularly military systems, such communication capability is unavailable for reasons of cost, reliability, bandwidth, survivability, security, and similar factors.

In practice, the sensors and associated local signal processors in distributed surveillance networks tend to be designed relatively independently of one another, with communication restricted to higher level signal characteristics (e.g., reports of emitter detections). While such practice has the virtue of simplicity, there is a potential loss in performance due to considering each sensor individually, rather than as an element in an overall surveillance system.

Motivated by such considerations, Tenney and Sandell formulated and solved a number of distributed detection problems (with random variable observations) in [5]. From a technical point-of-view, these problems were problems

*References are indicated by numbers in square brackets and appear at the end of this paper.

in team theory [6] or decentralized control theory [7]. Subsequently, a number of authors have generalized the formulations in [5] in various directions [8]-[12]; however, all have considered only the case of random variable observations.

In this paper, we generalize the distributed detection theory results of [5] to the case of waveform (i.e., stochastic process) observations. Thus we will have two hypotheses to test,

1. H^0: signal is absent, and
2. H^1: signal is present.

Corresponding to these two hypotheses, we have the potential decisions $u_i=0$ (H^0 is true) and $u_i=1$ (H^1 is true) made at the i-th local sensor. The cost of decision errors is measured by a cost function $J(u_1,u_2,H)$ (for the case of two sensors); it is desired to choose local decisions u_i on the basis of locally available observations $y_i(t)$ to minimize the expected value of J.

Note that in the special case of a surveillance system in which the decisions u_i are transmitted to a central fusion center (Fig. 1-1) where a final decision is made, the cost J has the special form

$$J(u_1,u_2,H) = J'(f(u_1,u_2),H) \tag{1-1}$$

where $J'(u,H)$ is the global decision cost and $f(u_1,u_2)$ is the (fixed) fusion law (e.g., voting) used to determine u from u_1 and u_2.

In the sequel, we will assume that the local signal processing implemented at the i-th sensor consists of the following two steps:*

*As will be noted subsequently, only in special cases is it in fact optimal to base the detection decision on the local likelihood ratio.

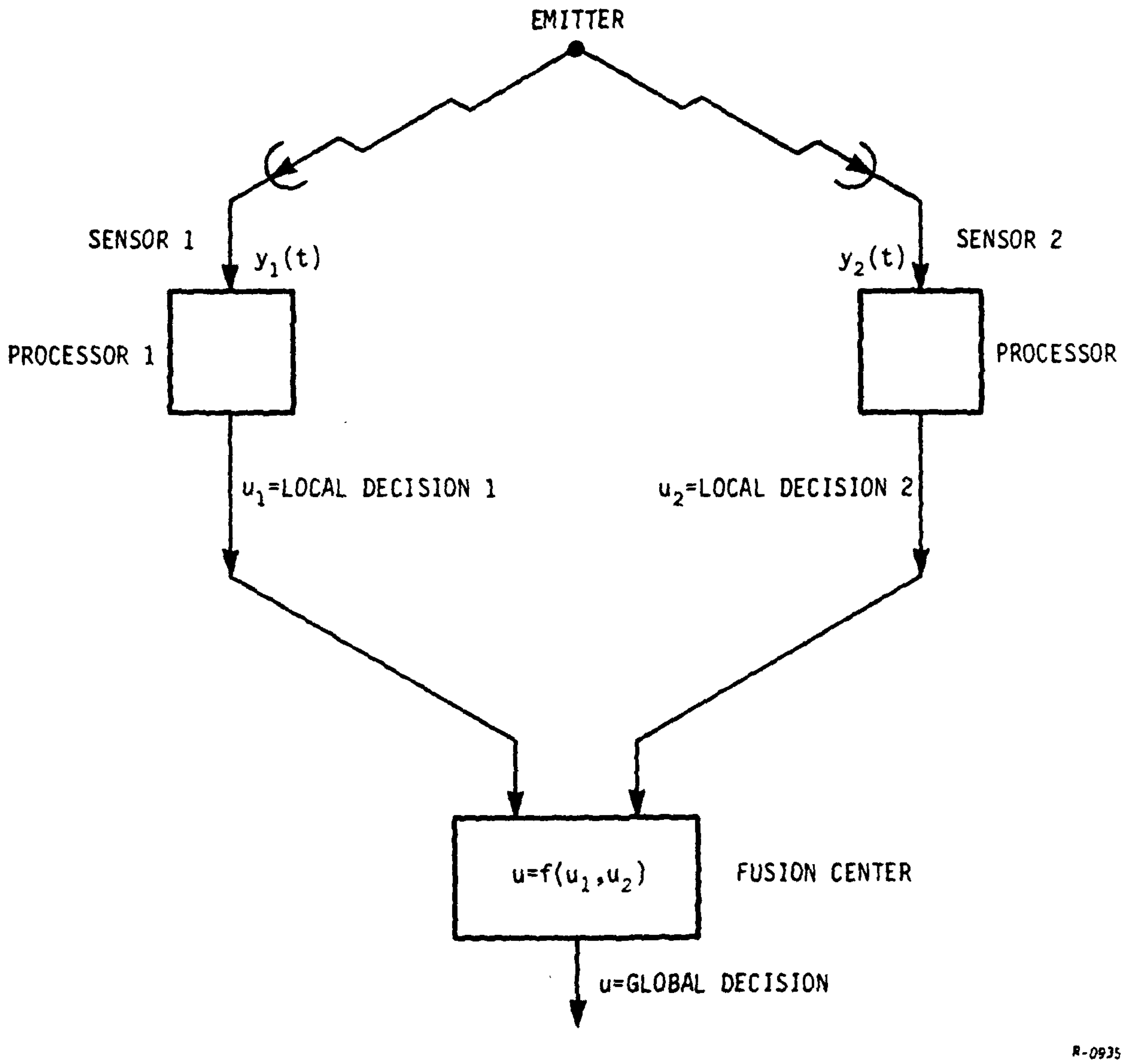

Figure 1-1. Surveillance System With Final Decision Made at a Centralized Decision Node.

1. Determine the local log-likelihood ratio ℓ_i; and

2. Implement the test*

$$\ell_i \underset{H_0}{\overset{H_1}{\gtrless}} T_i \quad . \tag{1-2}$$

We will then determine the optimal thresholds T_i.

In Section 2, we consider the case in which each $y_i(t)$ is either a known signal plus noise (H^1), or just noise (H^0). In the special cases in which: (1) the noise is uncorrelated between sensors, and (2) the noise is correlated between sensors but is white and the signals are linearly dependent, we are able to obtain sufficient statistics for local decisionmaking. The technique involves expanding the observation processes into a Karhuren-Loeve series. Numerical results are provided for the case of detection of a sinusoidal signal in white Gaussian noise. Numerical results are presented for both the general Bayesian case and for the case with a centralized fusion center.

In Section 3 we consider the case in which the signal is a Guassian stochastic process. Results are more difficult to obtain here, since the log-likelihood ratio is a nonlinear (quadratic) function of the observations even in the centralized case. However, in the important special case of an ideal band limited signal in white noise, we are able to obtain a closed form (albeit rather complex) expression for the joint probability density function of the local log-likelihood ratios. This expression permits us to solve for

*The notation means "choose H^1 if $\ell_i > T_i$, and choose H^0 if $\ell_i < T_i$."

the optimal distributed detection thresholds. Numerical results are presented for both the general Bayesian case and for the case with a central fusion center.

Section 4 contains a summary, some conclusions, and suggestions for future research.

SECTION 2

KNOWN SIGNAL IN NOISE

In this section we consider a distributed detection problem in which the signals are known deterministic waveforms and the noise processes are colored Gaussian processes. As stated earlier, we will restrict attention to the case of two detectors each of which must make a binary decision based on local observations.

2.1 PROBLEM FORMULATION AND PRELIMINARY ANALYSIS

We assume that there are two sensors, indexed by i=1,2 and that there are two hypotheses to be tested based on the sensor observations. The observations under the two hypotheses are modeled by the equations

$$\begin{aligned} H^1:\ & y_i(t) = \sqrt{E_i}\, s_i(t) + n_i(t) \qquad T_o<t<T_f \\ H^0:\ & y_i(t) = n_i(t) \qquad T_o<t<T_f \ . \end{aligned} \tag{2-1}$$

We assume that the $s_i(t)$ are known signals with unit energy and are zero outside the interval $[0,T]$ where $T_o<0<T<T_f$. The $n_i(t)$ are assumed to be zero-mean Gaussian processes where

$$E\{n_i(t)n_j(\tau)\} = K_{ij}^{n}(t,\tau) \quad , \quad i,j=1,2 \tag{2-2}$$

and we assume (to avoid the possibility of singular detection)

$$K_{ii}^{n}(t,\tau) = K_i^{c}(t,\tau) + N_i^{2}\delta(t-\tau) \tag{2-3}$$

with $N_i \neq 0$. Note that we do <u>not</u> assume that the noise processes are uncorrelated between sensors, i.e., we allow $K_{ij}^{n}(t,\tau)$ to be nonzero for $i \neq j$.

The approach we will use to determine the optimal distributed detection laws consists of expanding the received waveforms in a Karhunen-Loeve (K-L) expansion [1] and considering the problem formed by truncating the infinite series of K-L coefficients to the first K terms. The truncated problem can be approached via the technique of [5] and results for the waveform problem obtained by taking the limit as $K \to \infty$.

Thus we expand $y_i(t)$ via*

$$y_i(t) = \lim_{K\to\infty} \sum_{k=1}^{K} y_i^k \phi_i^k(t) \quad , \quad T_o < t < T_f \tag{2-4}$$

where the $\phi_i^k(t)$ satisfy

$$\lambda_i^k \phi_i^k(t) = \int_{T_o}^{T_f} K_i^{c}(t,u)\phi_i^k(u)du \quad , \quad T_o < t < T_f \quad . \tag{2-5}$$

Under H^1 we have

$$y_i^k = s_i^k + n_i^k \tag{2-6}$$

where

*$x = \lim_{k\to\infty} x_k$ is defined to mean $\lim_{k\to\infty} \| x - x_k \|^2 = 0$.

$$s_i^k \triangleq \int_{T_o}^{T_f} \sqrt{E_i}\, s_i(t)\phi_i^k(t)dt \tag{2-7}$$

$$n_i^k \triangleq \int_{T_o}^{T_f} n_i(t)\phi_i^k(t)dt \quad , \tag{2-8}$$

while under H^0 we have that

$$y_i^k = n_i^k \quad . \tag{2-9}$$

It is straightforward to show that

$$E\{n_i^k\} = 0 \tag{2-10}$$

$$E\{(n_i^k)^2\} = \lambda_i^k + N_i^2 \tag{2-11}$$

and that the K-L coefficients corresponding to each sensor are uncorrelated:

$$E\{n_i^\ell n_i^k\} = 0 \qquad \ell \neq k \quad . \tag{2-12}$$

However, the K-L coefficients for the two sensors are correlated, that is,

$$E\{(y_1^\ell - E\{y_1^\ell | H^j\})(y_2^k - E\{y_2^k | H^j\} | H^j\} = \int_{T_o}^{T_f}\int_{T_o}^{T_f} \phi_1^\ell(t)K_{12}^n(t,u)\phi_2^k(u)dtdu \triangleq c^{\ell k} \quad ,$$

$$j=0,1 \quad , \quad \ell \neq k \tag{2-13}$$

which is not zero in general.

Using the results of [5] we can determine (implicitly) the optimal distributed decision law for the problem based on the first K coefficients of the K-L expansion. Define

$$\underline{y}_i^K = [y_i^1, y_i^2, \ldots, y_i^K] \tag{2-14}$$

and

$$\Lambda_i(\underline{y}_i^K) = \prod_{k=1}^{K} \exp\left[- \frac{(s_i^k)^2 - 2y_i^k s_i^k}{2(\lambda_i^k + N_i^2)} \right] . \tag{2-15}$$

The optimal decision law is then given by the solution of the following two equations:

$$\Lambda_1(\underline{y}_1^K) \underset{1}{\overset{0}{\gtrless}} \frac{\Delta J_1 + \Delta J_2 \displaystyle\int \cdots \int_{\underline{y}_2^K | u_2 = 0} p(\underline{y}_2^K | \underline{y}_1^K, H^1) d\underline{y}_2^K}{\Delta J_3 + \Delta J_4 \displaystyle\int \cdots \int_{\underline{y}_2^K | u_2 = 0} p(\underline{y}_2^K | y_1^K, H^0) d\underline{y}_2^K} \tag{2-16a}$$

$$\Lambda_2(\underline{y}_2^K) \underset{1}{\overset{0}{\gtrless}} \frac{\Delta J_5 + \Delta J_2 \displaystyle\int \cdots \int_{\underline{y}_1^K | u_1 = 0} p(\underline{y}_1^K | \underline{y}_2^K, H^1) d\underline{y}_1^K}{\Delta J_6 + \Delta J_4 \displaystyle\int \cdots \int_{\underline{y}_1^K | u_1 = 0} p(\underline{y}_1^K | y_2^K, H^0) d\underline{y}_2^K} , \tag{2-16b}$$

where the ΔJ_n are given by the following equations:

$$\Delta J_1 = J(0,1,H^1) - J(1,1,H^1) \tag{2-17a}$$

$$\Delta J_2 = J(0,0,H^1) + J(1,1,H^1) - J(1,0,H^1) - J(0,1,H^1) \tag{2-17b}$$

$$\Delta J_3 = J(1,1,H^0) - J(0,1,H^0) \tag{2-17c}$$

$$\Delta J_4 = J(1,0,H^0) + J(0,1,H^0) - J(0,0,H^0) - J(1,1,H^0) \tag{2-17d}$$

$$\Delta J_5 = J(1,0,H^1) - J(1,1,H^1) \tag{2-17e}$$

and

$$\Delta J_6 = J(1,1,H^0) - J(1,0,H^0) \quad . \tag{2-17f}$$

Note that the decision law for each sensor is required to determine the region of integration for the right-hand side of Eq. 2-16. Thus, the determination of the optimal distributed decision law requires the solution of coupled nonlinear functional equations. No general analytic solution to these equations has been determined and numerical techniques do not seem to be computationally feasible. In general, it is necessary to assume some special form for the local signal processing (e.g., a likelihood ratio computation) and optimize the thresholds. However, in certain special cases more can be said.

2.2 SPECIAL CASES

2.2.1 Uncorrelated Noise

The easiest case to consider is that in which the noise is uncorrelated between sensors, i.e., $K^n_{12}(t,\tau)\equiv 0$. Under this assumption we have that

$$p(\underline{y}_i^K|\underline{y}_j^K,H^\ell) \equiv p(\underline{y}_i^K|H^\ell) \qquad i,j=1,2,i\neq j,\ell=0,1 \tag{2-18}$$

and thus Eq. 2-16 becomes

$$\Lambda_1(\underline{y}_1^K) \underset{1}{\overset{0}{\gtrless}} \frac{\Delta J_1+\Delta J_2 \displaystyle\int_{\underline{y}_2^K|u_2=0}\cdots\int p(\underline{y}_2^K|H^1)d\underline{y}_2^K}{\Delta J_3+\Delta J_4 \displaystyle\int_{\underline{y}_2^K|u_2=0}\cdots\int p(\underline{y}_2^K|H^0)d\underline{y}_2^K} \triangleq T_1^K \tag{2-19}$$

$$\Lambda_2(\underline{y}_2^K) \underset{1}{\overset{0}{\gtrless}} \frac{\Delta J_5+\Delta J_2 \displaystyle\int_{\underline{y}_1^K|u_1=0}\cdots\int p(\underline{y}_1^K|H^1)d\underline{y}_1^K}{\Delta J_3+\Delta J_4 \displaystyle\int_{\underline{y}_1^K|u_1=0}\cdots\int p(\underline{y}_1^K|H^0)d\underline{y}_1^K} \triangleq T_2^K \; . \tag{2-20}$$

Note that the right-hand sides of the above equations do not depend upon the local observations, but are constants, which we define as T_i^K. Thus, the optimal distributed decision law is given by a pair of local likelihood ratio tests where the thresholds are implicitly defined by Eqs. 2-19 and 2-20.

If we take the limit as $K\to\infty$ then one can show (see [1]) that the optimal decision laws can be written as

$$\int_{T_o}^{T_f} y_i(t)g_i(t)dt \underset{0}{\overset{1}{\gtrless}} \frac{1}{2}\sqrt{E_i} \int_{T_o}^{T_f} s_i(t)g_i(t)dt - \tau_i \tag{2-21}$$

where the left-hand side of Eq. 2-21 is determined by

$$g_i(t) = \sqrt{E_i} \int_{T_o}^{T_f} Q_i(t,u)s_i(u)du \tag{2-22}$$

and $Q_i(t,u)$ is defined by

$$\int_{T_o}^{T_f} K_{ii}^{n}(t,u)Q_i(u,v)du = \delta(t-v) \quad . \tag{2-23}$$

The right-hand side of Eq. 2-21 is a constant (observation-independent) threshold and the τ_i satisfy the following nonlinear coupled <u>algebraic</u> equations:

$$\tau_1 = \ell n \left\{ \frac{\Delta J_1 + \Delta J_2 [1-\mathrm{erf}((\tau_2 + m_2)/\sqrt{2m_2})]}{\Delta J_3 + \Delta J_4 [1-\mathrm{erf}((\tau_2 - m_2)/\sqrt{2m_2})]} \right\} \frac{p^1}{p^0} \tag{2-24}$$

$$\tau_2 = \ell n \left\{ \frac{\Delta J_5 + \Delta J_2 [1-\mathrm{erf}((\tau_1 + m_1)/\sqrt{2m_1})]}{\Delta J_6 + \Delta J_4 [1-\mathrm{erf}((\tau_1 - m_1)/\sqrt{2m_1})]} \right\} \frac{p^1}{p^0} \tag{2-25}$$

where

$$m_i \triangleq \frac{\sqrt{E_i}}{2} \int_{T_o}^{T_f} s_i(t)g_i(t)dt \quad . \tag{2-26}$$

Note that in general, Eqs. 2-24 and 2-25 do not have a unique solution and thus the optimal decision law must be determined by computing all solutions and then comparing their performance. The decision law of Eq. 2-21 extends the results of [5] to the case of waveform observations and this law can be interpreted as a pair of local likelihood ratio tests with thresholds which are jointly optimized according to a system-wide measure of performance.

2.2.2 Linearly Dependent Signals and Correlated White Noise

In this special case we assume that $s_1 = s_2 \triangleq s$ so that the observations are given by

$$\begin{aligned} H^1: \quad & y_i(t) = \sqrt{E_i}\, s(t) + n_i(t) \qquad 0<t<T \\ H^0: \quad & y_i(t) = n_i(t) \qquad 0<t<T \quad . \end{aligned} \tag{2-27}$$

Furthermore we assume that the $n_i(t)$ are zero-mean unit spectral height, white Gaussian noise processes where

$$E\{n_1(t)n_2(\tau)\} = \rho\delta(t-\tau) \quad . \tag{2-28}$$

As usual we expand the received waveform in a K-L expansion where now we choose the $\phi_i^k(t)$ to be any complete orthonormal set such that

$$\phi_1^k = \phi_2^k(t) = \phi^k(t) \qquad 0<t<T \tag{2-29}$$

and

$$\phi^1(t) = s(t) \qquad 0<t<T \quad . \tag{2-30}$$

It is easy to verify that if

$$y_i(t) = \lim_{K\to\infty} \sum_{k=1}^{K} y_i^k \phi^k(t) \tag{2-31}$$

then

$$y_i^1 = \begin{cases} \sqrt{E_i} + w_i & : H^1 \\ w_i & : H^0 \end{cases} \tag{2-32}$$

$$y_i^k = \begin{cases} w_i & : H^1 \\ w_i & : H^0 \end{cases} \quad k=2,\ldots \tag{2-33}$$

where

$$w_i \triangleq \int_0^T \phi^i(t) n_i(t) dt$$

and

$$E\{w_1^k w_2^\ell\} = \rho\delta(k-\ell) \quad . \tag{2-34}$$

Note that the coefficients y_i^k for $k\geq 2$ have the same statistics under both hypotheses and are independent of the y_i^1. Thus any optimal decision law will use only the y_i^1 to decide between the hypotheses H^0 and H^1. The y_i^1 are thus scalar sufficient statistics for optimal decisionmaking in the team decision problem with information pattern defined by Eqs. 2-27 and 2-28.

Any optimal decision law can therefore be specified for this problem by defining the regions of the real line in which y_i^1 must lie for a decision of H^1 to be made. These regions can be specified by their endpoints, and thus the decision law can be characterized by a set of endpoints, or thresholds,

for each detector. The optimal distributed decision law is determined by choosing collections of thresholds $\{[T_i^{2\ell-1}, T_i^{2\ell}]\}_{\ell=1}^{L_i}$, i=1,2 as follows:

$$u_i = \begin{cases} 1 & \text{if } T_i^{2\ell} < \int_0^T s_i(t)y_i(t)dt < T_i^{2\ell+1}, \ \ell=1,\ldots,L \\ 0 & \text{otherwise} \end{cases} \quad . \qquad (2\text{-}35)$$

Necessary conditions for the optimality of the thresholds are readily developed if L_i is specified. Since the number L_i of such thresholds is arbitrary one must compare the performance of laws with different L_i's to determine the optimal distributed detection law. We have calculated optimal laws based on the assumption that $L_1=L_2=L=1$ or 3.* In all cases the laws based on the assumption that L=1 proved superior. While we have been unable to prove that L=1 is optimal, we will only consider that case in the sequel.†

The optimal detection law for the L=1 case is given by

$$\int_0^T s(t)y_i(t)dt \underset{0}{\overset{1}{\gtrless}} T_i \qquad (2\text{-}36)$$

wherre T_1 and T_2 satisfy

*Clearly $u_i=0$ as $y_i^1\to-\infty$ and $u_i=1$ as $y_i^1\to+\infty$ so that an even number of intervals is required and thus L must be odd.

†Note that the necessary conditions for optimality consist of 2L nonlinear coupled equations; the assumption of L=1 leads to considerable reduction in the computational burden.

$$T_1 = \frac{\sqrt{E_1}}{2} - \frac{1}{\sqrt{E_1}} \ln \left\{ \frac{\Delta J_1 + \Delta J_2 \ \mathrm{erf}[(T_2 - \sqrt{E_2} - \rho(T_1 - \sqrt{E_1}))/\sqrt{1-\rho^2}}{\Delta J_3 + \Delta J_4 \ \mathrm{erf}[(T_2 - \rho T_1)/\sqrt{1-\rho^2}]} \right\} \tag{2-37}$$

$$T_2 = \frac{\sqrt{E_2}}{2} - \frac{1}{\sqrt{E_2}} \ln \left\{ \frac{\Delta J_5 + \Delta J_2 \ \mathrm{erf}[(T_1 - \sqrt{E_1} - \rho(T_2 - \sqrt{E_2}))/\sqrt{1-\rho^2}}{\Delta J_6 + \Delta J_4 \ \mathrm{erf}[(T_1 - \rho T_2)/\sqrt{1-\rho^2}]} \right\} . \tag{2-38}$$

For the special cases considered above the "local" likelihood ratio is a sufficient statistic for optimal distributed decisionmaking. More generally, coupled functional equations must be solved to determine the optimal distributed detection law. However, given the difficulty of solving such equations and the above results, a reasonable choice for a distributed detection law is to use local likelihood ratio tests with globally optimized thresholds. Note that this choice implies that the same signal processing can be used whether a sensor is used alone or as part of a surveillance network - a strong practical requirement! In Section 3 we will take this approach when considering the detection of an unknown signal in noise.

2.3 NUMERICAL EXAMPLES

Here we consider an example involving a scalar observation with correlated noise.* First we introduce the observation model and rewrite the necessary conditions, then in subsections 2.3.1 and 2.3.2 we consider two different global cost functions.

*The special case of uncorrelated noise reduces to the scalar observation problem studied in [5] and thus we do not consider it here.

Assume that the observations are given by

$$H^1: \quad y_i(t) = \sqrt{2E_i}\,\sin(2\pi t) + n_i(t) \qquad 0<t<1$$

$$H^0: \quad y_i(t) = n_i(t) \qquad 0<t<1 \tag{2-39}$$

where $n_i(t)$ is zero-mean unit variance white Gaussian noise with

$$E\{n_1(t)n_2(\tau)\} = \rho\delta(t-\tau) \quad . \tag{2-40}$$

We assume $p^0=p^1=1/2$ and note that the detection law can thus be written as

$$y_i^1 \triangleq \sqrt{2} \int_0^T y_i(t)\sin(2\pi t)dt \underset{0}{\overset{1}{\gtrless}} T_i \tag{2-41}$$

where the T_i are solutions to Eqs. 2-37 and 2-38.

2.3.1 Bayesian Formulation

We assume that the thresholds T_i are to be selected so as to minimize the expected Bayes cost where

$$J = \begin{cases} 0 & \text{if} \quad u_1=u_2=H \\ 1 & \text{if} \quad u_1\neq u_2 \\ k & \text{if} \quad u_1=u_2\neq H \end{cases} \quad . \tag{2-42}$$

This type of cost criterion arises in situations where it is not precisely twice as costly to make two errors as it is to make one. For example, if weapons are collocated with sensors and targets are automatically attacked when detected, then having two missed detections may be much more serious than

having one missed detection. This situation would be modeled by choosing $k>2$, so that a double error is more than twice as costly as two single errors.

For J defined by Eq. 2-42, Eqs. 2-37 and 2-38 become

$$T_1 = \frac{\sqrt{E_1}}{2} - \frac{1}{\sqrt{E_1}} \ln \left\{ \frac{1+(k-2)\operatorname{erf}\left[\dfrac{T_2-\sqrt{E_2}-\rho(T_1-\sqrt{E_1})}{\sqrt{1-\rho^2}}\right]}{(k-1)-(k-2)\operatorname{erf}\left[\dfrac{T_2-\rho T_1}{\sqrt{1-\rho^2}}\right]} \right\} \tag{2-43}$$

$$T_2 = \frac{\sqrt{E_2}}{2} - \frac{1}{\sqrt{E_2}} \ln \left\{ \frac{1+(k-2)\operatorname{erf}\left[\dfrac{T_1-\sqrt{E_1}-\rho(T_2-\sqrt{E_2})}{\sqrt{1-\rho^2}}\right]}{(k-1)-(k-2)\operatorname{erf}\left[\dfrac{T_1-\rho T_2}{\sqrt{1-\rho^2}}\right]} \right\} \tag{2-44}$$

and we note that the thresholds for the locally optimal test* ($T_1=\sqrt{E_1}/2$, $T_2=\sqrt{E_2}/2$) satisfy Eqs. 2-43 and 2-44. This can be seen by noting that

*That is the test with thresholds that would be optimal for each sensor considered in isolation when a minimum probability of error criterion is used.

$$\ell n \left\{ \frac{1+(k-2)\operatorname{erf}\left[-\dfrac{\sqrt{E_2}-\rho\sqrt{E_1}}{2\sqrt{1-\rho^2}} \right]}{(k-1)-(k-2)\operatorname{erf}\left[\dfrac{\sqrt{E_2}-\rho\sqrt{E_1}}{2\sqrt{1-\rho^2}} \right]} \right\} = \ell n \left\{ \frac{1+(k-2)\left(1-\operatorname{erf}\left[\dfrac{\sqrt{E_2}-\rho\sqrt{E_1}}{2\sqrt{1-\rho^2}} \right] \right)}{(k-1)-(k-2)\operatorname{erf}\left[\dfrac{\sqrt{E_2}-\rho\sqrt{E_1}}{2\sqrt{1-\rho^2}} \right]} \right\}$$

$$= \ell n \left\{ \frac{(k-1)-(k-2)\operatorname{erf}\left[\dfrac{\sqrt{E_2}-\rho\sqrt{E_1}}{2\sqrt{1-\rho^2}} \right]}{(k-1)-(k-2)\operatorname{erf}\left[\dfrac{\sqrt{E_2}-\rho\sqrt{E_1}}{2\sqrt{1-\rho^2}} \right]} \right\} .$$

$$= \ell n(1) = 0 \quad . \tag{2-45}$$

Equations 2-43 and 2-44 are symmetric; if $(\hat{T}_1,\hat{T}_2)$ is a solution, then $(\sqrt{E_1}-\hat{T}_1, \sqrt{E_2}-\hat{T}_2)$ is also a solution. This is seen by noting that if

$$\hat{T}_1 = \frac{\sqrt{E_1}}{2} - \frac{1}{\sqrt{E_1}} \ell n \left\{ \frac{1+(k-2)\operatorname{erf}\left[\dfrac{\hat{T}_2-\sqrt{E_2}-\rho(\hat{T}_1-\sqrt{E_1}}{\sqrt{1-\rho^2}} \right]}{(k-1)-(k-2)\operatorname{erf}\left[\dfrac{\hat{T}_2-\rho\hat{T}_1}{\sqrt{1-\rho^2}} \right]} \right\} \tag{2-46}$$

then

$$\sqrt{E_1}-\hat{T}_1$$

$$= \frac{\sqrt{E_1}}{2} - \frac{1}{\sqrt{E_1}} \ln \left\{ \frac{(k-1)-(k-2)\mathrm{erf}\left[(\hat{T}_2-\rho\hat{T}_1)/\sqrt{1-\rho^2}\right]}{1+(k-2)\mathrm{erf}\left[(\hat{T}_2-\sqrt{E_2}-\rho(\hat{T}_1-\sqrt{E_1}))/\sqrt{1-\rho^2}\right]} \right\}$$

$$= \frac{\sqrt{E_1}}{2} - \frac{1}{\sqrt{E_1}} \ln \left\{ \frac{(k-1)-(k-2)\left[1-\mathrm{erf}\left[\frac{(\sqrt{E_2}-\hat{T}_2)-\sqrt{E_2}-\rho\left((\sqrt{E_1}-\hat{T}_1)-\sqrt{E_1}\right)}{\sqrt{1-\rho^2}}\right]\right]}{1+(k-2)\left[1-\mathrm{erf}\left[(\sqrt{E_2}-\hat{T}_2)-\rho(\sqrt{E_1}-\hat{T}_1)/\sqrt{1-\rho^2}\right]\right]} \right\}$$

$$= \frac{\sqrt{E_1}}{2} - \frac{1}{\sqrt{E_1}} \ln \left\{ \frac{1+(k-2)\mathrm{erf}\left[\frac{(\sqrt{E_2}-\hat{T}_2)-\sqrt{E_2}-\rho(\sqrt{E_1}-\hat{T}_1)-\sqrt{E_1})}{\sqrt{1-\rho^2}}\right]}{(k-1)-(k-2)\mathrm{erf}\left[\frac{(\sqrt{E_2}-\hat{T}_2)-\rho(\sqrt{E_1}-\hat{T}_1)}{\sqrt{1-\rho^2}}\right]} \right\} . \tag{2-47}$$

The costs associated with these two symmetric solutions are identical and thus, if a solution with $T_i \neq \sqrt{E_i}/2$ exists, only one of this pair need be evaluated.

Graphical solution of Eqs. 2-37 and 2-38 shows that there are at most three solutions to these equations, and for certain values of k and ρ there is only one solution (the "locally optimal test" (LOT)). In Figs. 2-1 and 2-2 we plot the Bayes cost associated with the decentralized likelihood ratio test (DLRT) and with the LOT solutions for $\sqrt{E_1}=1$, $\sqrt{E_2}=2$ and various ρ and k.

Figure 2-1 is a plot of the Bayes cost as a function of k for $\rho=0$ and $\rho=0.5$. Note that, for larger ρ, the optimal DLRT solution can be much better than the LOT. This is because the optimal solution skews the thresholds to avoid the double errors which are more common for larger ρ and more costly for larger k.

Note that for the LOT the cost increases as an affine function of k since the probability of a double error is independent of k. The expected cost for the DLRT never exceeds 1 since that value can be obtained by a suboptimal law with thresholds set so that one detector always decides 0 while the other always decides 1.

Figure 2-2 is a plot of the Bayes cost as a function of ρ for k=5. Again we see that as ρ increases the DLRT becomes much better than the LOT. This occurs because, as $\rho \to 1$, the probability of a double error (for fixed thresholds) increases. The DLRT solution skews the thresholds to decrease the probability of a double error. This yields a 27 percent decrease in cost over the LOT.

These results indicate that in some cases there is a significant gain to be had by using the optimal decentralized likelihood ratio test rather than a naive approach which ignores the correlation between sensors.

2.3.2 Surveillance System Design

In this subsection we assume that a local decision u_i based only on the information provided by sensor i is sent to a fusion center where a global decision u is made. We assume that the fusion center decision rule is defined as u=1 if and only if $u_1=u_2=1$. We display the overall performance of such a surveillance system via a generalized network receiver operating

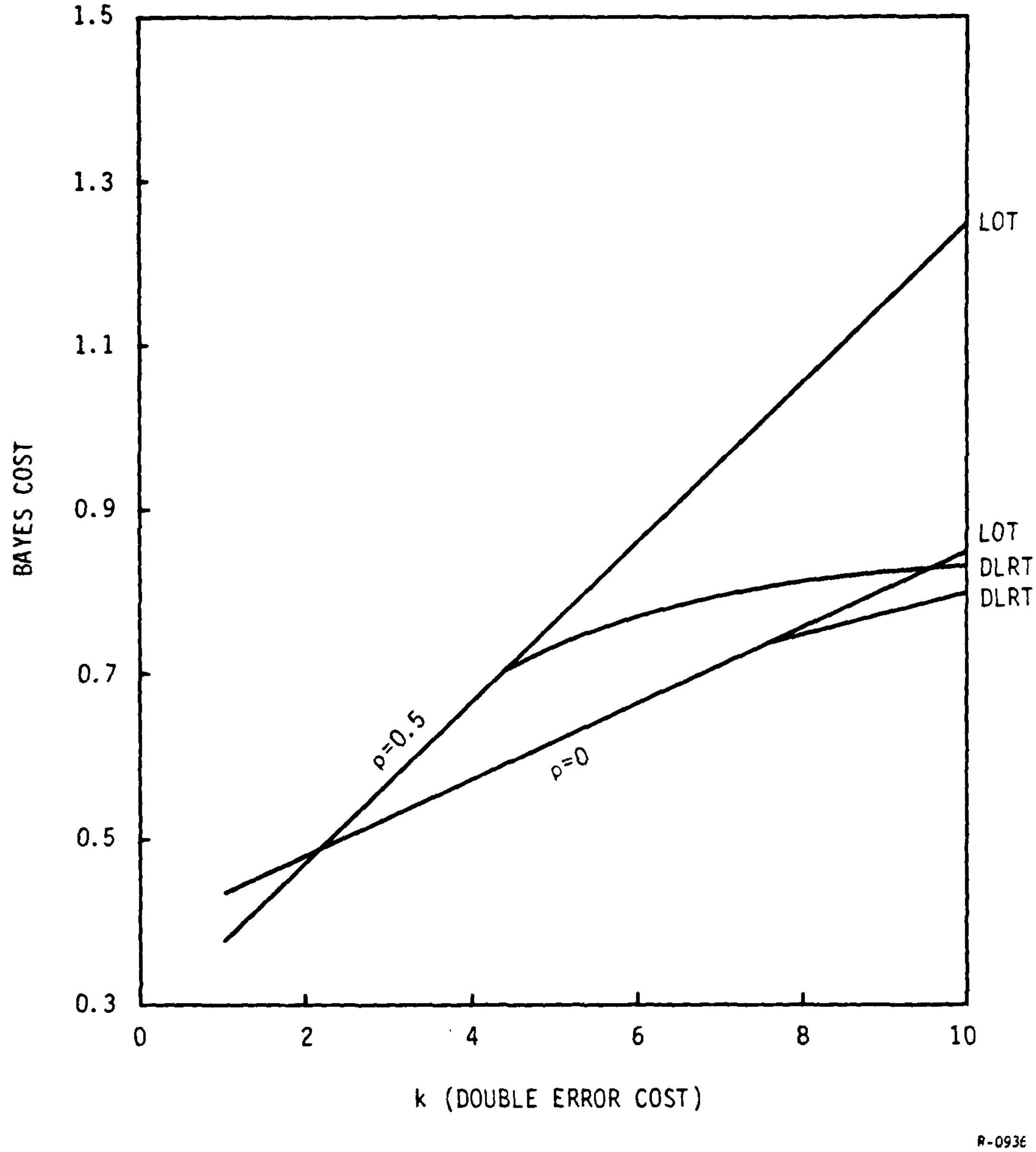

Figure 2-1. Bayes Cost as a Function of k for DLRT and LOT With $E_1=1$, $E_2=4$, and $\rho=0$ or 0.5.

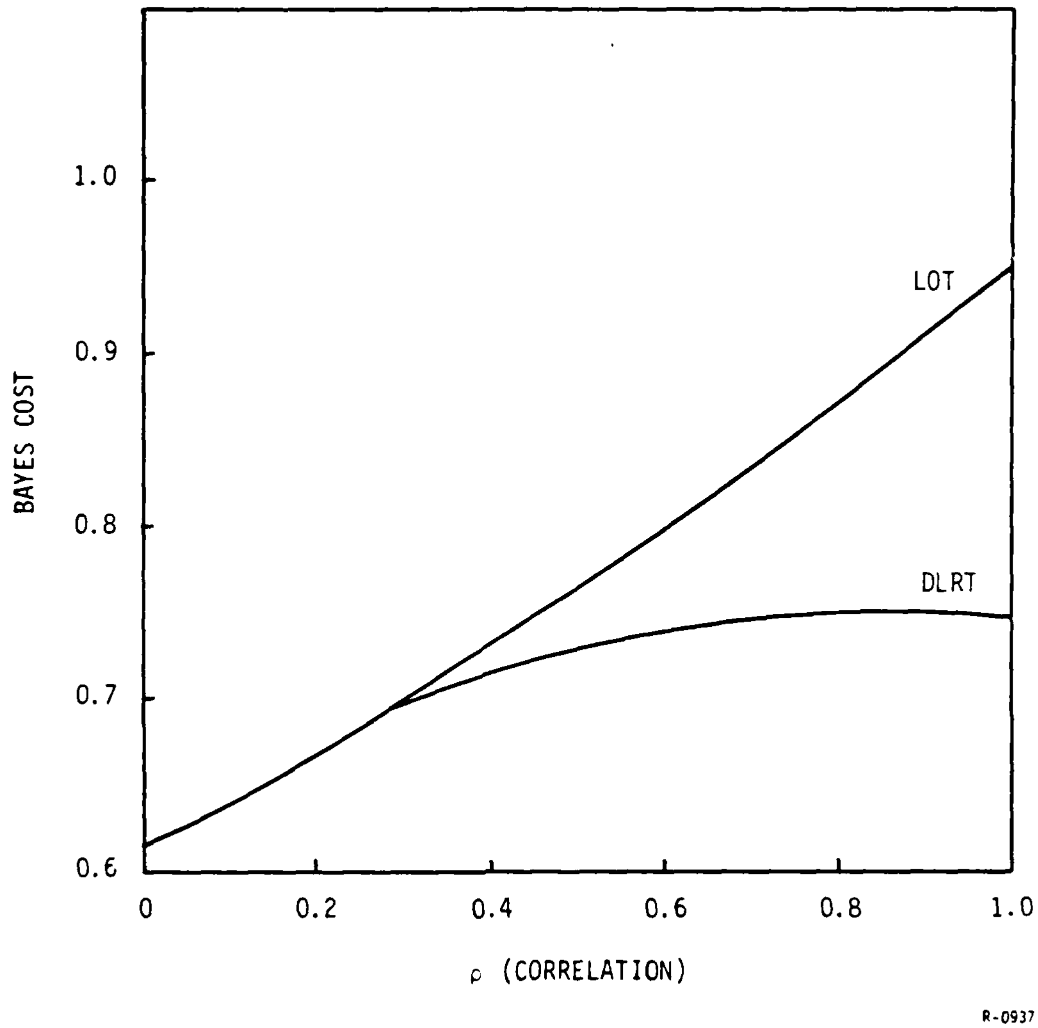

Figure 2-2. Bayes Cost as a Function of ρ for DLRT and LOT With $E_1=1$, $E_2=4$, and $k=5$.

characteristic (ROC) curve, which is a plot of the surveillance network's probability of detection versus probability of false alarm. In addition to plotting the performance of the DLRT we also plot the performance of the optimal centralized detection law. This allows us to determine how the performance of the surveillance system is degraded by requiring that only local decisions (processed information) rather than raw sensor data be transmitted to the fusion center.

The ROC can be obtained by varying the ratio of the cost of a false alarm to the cost of a missed detection. If we let the false alarm cost be unity and the missed detection cost be α then the necessary conditions for optimality become

$$T_1 = \frac{\sqrt{E_1}}{2} - \frac{1}{\sqrt{E_1}} \ln \left\{ \alpha \frac{1-\operatorname{erf}\left[(T_1-\sqrt{E_1}-\rho(T_2-\sqrt{E_2}))/\sqrt{1-\rho^2}\right]}{1-\operatorname{erf}\left[(T_1-\rho T_2)/\sqrt{1-\rho^2}\right]} \right\} \tag{2-48}$$

$$T_2 = \frac{\sqrt{E_2}}{2} - \frac{1}{\sqrt{E_2}} \ln \left\{ \alpha \frac{1-\operatorname{erf}\left[(T_2-\sqrt{E_2}-\rho(T_1-\sqrt{E_1}))/\sqrt{1-\rho^2}\right]}{1-\operatorname{erf}\left[(T_2-\rho T_1)/\sqrt{1-\rho^2}\right]} \right\} . \tag{2-49}$$

Figure 2-3 depicts the performance of the optimal centralized test and that of the DLRT for the case where the sensors are identical, $\sqrt{E_1}=\sqrt{E_2}=1$. We see that as ρ increases the centralized and DLRT results become more and more similar. Figure 2-3 also shows that as ρ increases the performance degrades.

This occurs in the centralized problem because as ρ increases the information available for decisionmaking effectively decreases from 2 independent observations with ρ=0 to one observation with ρ=1. As ρ increases in the DLRT case, each sensor has a better and better indication of what the other observation was, and thus the centralized solution can be more closely approximated by the decentralized solution.

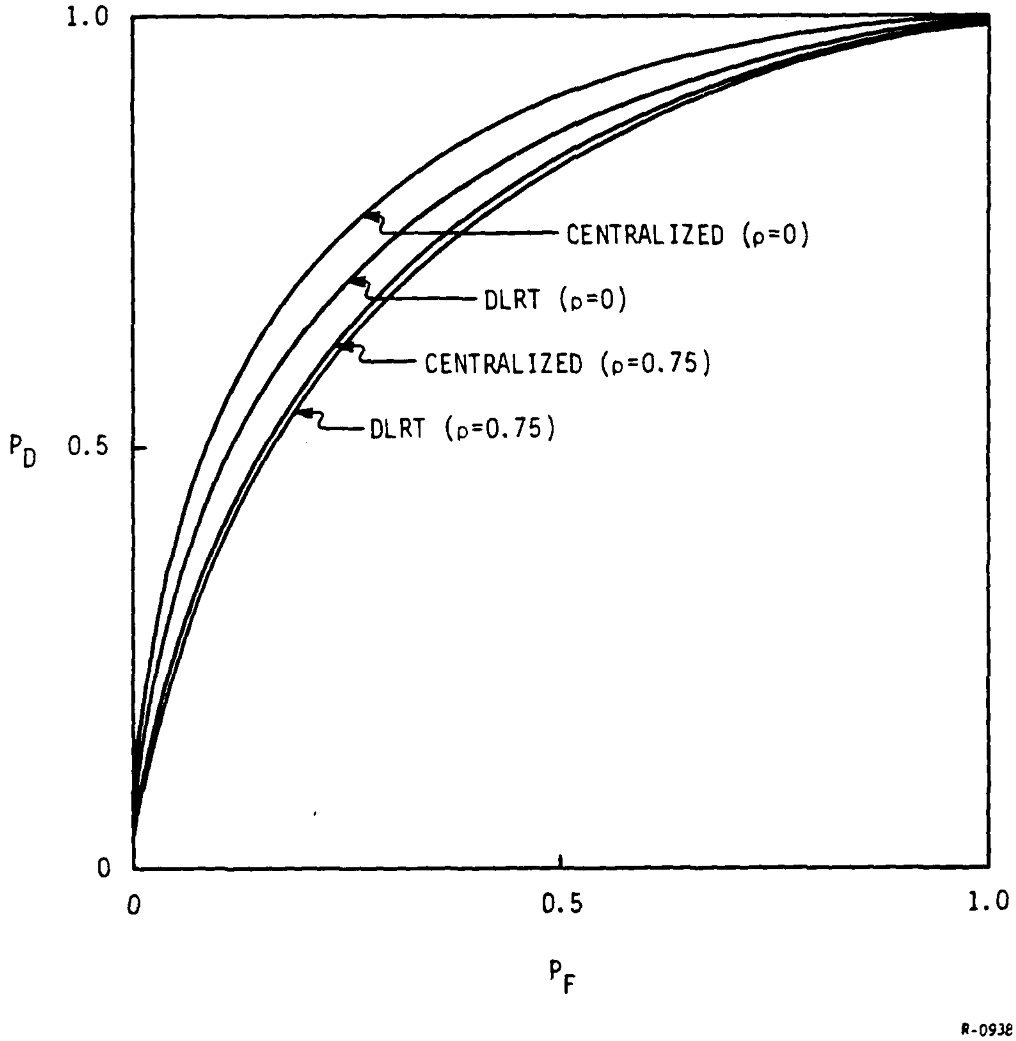

Figure 2-3. ROCs for Centralized Test and DLRT With $E_1=1$, $E_2=1$, and ρ=0.0 and 0.75.

Note, however, that the performance of the DLRT, while more closely approaching that of the centralized test, degrades as $\rho \to 1$. This can be understood by noting that if p is the probability of a local decision being wrong then the probability of a double error is approximately p^2 when $\rho=0$ but is p when $\rho=1$. Since the fusion center always makes an incorrect decision when both local decisions are wrong, the performance degrades as $\rho \to 1$.

The phenomenon of the DLRT and centralized results growing closer together as ρ increases is not universal. Figure 2-4 illustrates the behavior of these two decision laws for the case of asymmetric sensors, $\sqrt{E_1}=1$ and $\sqrt{E_2}=2$. We see that for the DLRT, the performance degrades as $\rho \to 1$. The reason is exactly the same as in the case of $\sqrt{E_1}=\sqrt{E_2}=1$. For the centralized case however, the performance becomes perfect as $\rho \to 1$. This occurs because by differencing the sensor observations one has

$$H^1: \quad \Delta y(t) = y_1(t) - y_2(t) = (\sqrt{2}-1)\sin(2\pi t) + n_1(t) - n_2(t) \qquad (2\text{-}50)$$

$$H^0: \quad \Delta y(t) = y_1(t) - y_2(t) = n_1(t) - n_2(t) \quad . \qquad (2\text{-}51)$$

As $\rho \to 1$, $\Delta y \to (\sqrt{2}-1)\sin(2\pi t)$ if H^1 is true and $\Delta y \to 0$ if H^0 is true, thus perfect detection is possible.

These graphs illustrate that the performance difference between a DLRT and a centralized test is strongly dependent on the problem being considered. Similarly, the benefit in reduction of communications requirements achieved by using a DLRT rather than a centralized test is highly problem dependent. It is thus the case that detailed analysis is required to determine whether a

DLRT or a centralized test should be implemented in a given situation. The theory developed in this paper helps to provide the basis for developing the tradeoffs.

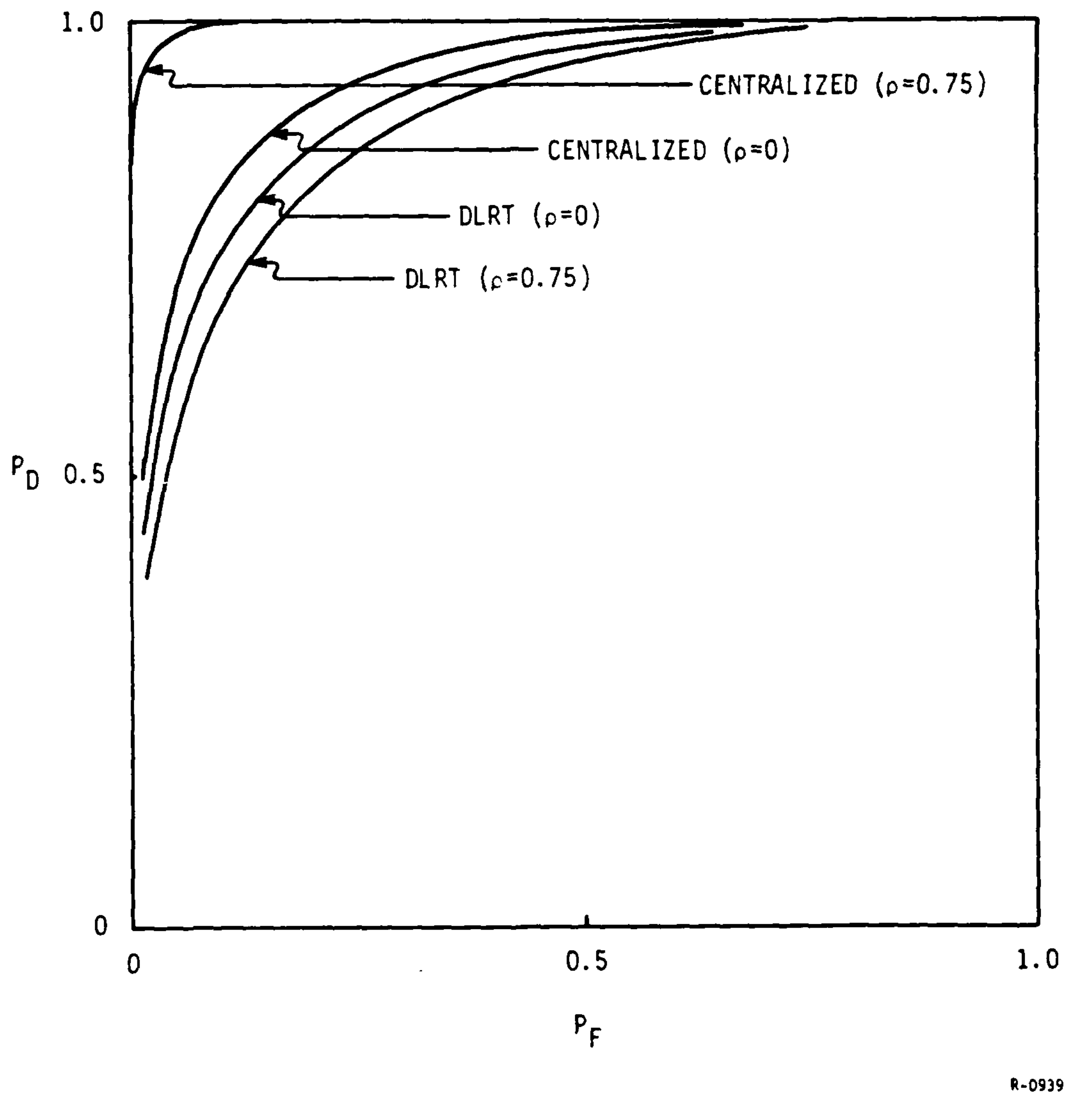

Figure 2-4. ROCs for Centralized Test and DLRT With $E_1=1$, $E_2=4$, and $\rho=0.0$.

SECTION 3

UNKNOWN SIGNAL IN NOISE

In this section we consider a distributed detection problem in which the signal is a random Gaussian process and the noises are white Gaussian processes. Again we consider only the case of two detectors and binary hypotheses.

We assume that the i-th sensor's observation under the two hypotheses is modeled by:

$$\begin{aligned} H^1: \quad & y_i(t) = c_i s_i(t) + n_i(t) \qquad 0<t<T \\ H^0: \quad & y_i(t) = n_i(t) \qquad 0<t<T \end{aligned} \tag{3-1}$$

where the $n_i(t)$ are independent zero-mean unit spectral height, white Gaussian noise processes and the signals $s_i(t)$ are zero-mean unit power Gaussian processes with known covariances:

$$E\{s_i(t)s_j(\tau)\} \triangleq K_{ij}^{s}(t,\tau) \qquad 0<t,\tau<T \quad . \tag{3-2}$$

If we expand the $y_i(t)$ in K-L series (generated by $K_{ii}^{s}(t,\tau)$) then the optimal distributed detection test is given by Eqs. 2-16 and 2-17, where the $\Lambda_i(y_i)$ are now quadratic functions of the observations $y_i(t)$. Unlike the known signal in noise case, all the K-L coefficients have statistics which

depend upon which hypothesis is true. Thus none of the simplifications possible in subsection 2.2 can be applied - to determine the optimal detection law, coupled nonlinear functional equations must be solved.

Since we cannot solve the equations defining the optimal distributed detection test we look instead for a good alternative. The test we consider is that motivated by the previous section: the distributed likelihood ratio test (DLRT). These tests are defined by the equation

$$\ell_i \underset{0}{\overset{1}{\gtrless}} T_i \quad , \quad i=1,2 \tag{3-3}$$

where ℓ_i is the (local) log-likelihood ratio and the T_i are optimized for the best global surveillance system performance. This class of tests is not easy to analyze for the problem of detecting an unknown signal in noise. The difficulty arises because the local log-likelihood ratios are not Gaussian. In the discussion below we consider a problem for which we are able to obtain results.

3.1 IDEAL BANDLIMITED SIGNAL

Consider a problem in which the observations are modeled by*

$$\begin{aligned} H^1: \quad & y_i(t) = c_i s(t) + n_i(t) \qquad 0<t<T \\ H^0: \quad & y_i(t) = n_i(t) \qquad 0<t<T \end{aligned} \tag{3-4}$$

*The bandpass version of this problem is readily treated in an entirely analogous fashion; such a formulation is a reasonable model for passive detection of a radio transmitter operating at a known frequency and with a known bandwidth.

where s(t) is a zero-mean Gaussian stochastic process of unit power with an ideal bandlimited spectrum (Fig. 3-1) and the $n_i(t)$ are zero-mean unit spectral height, Gaussian white noise processes. We denote the covariance of the signal by $K^s(t,\tau)$.

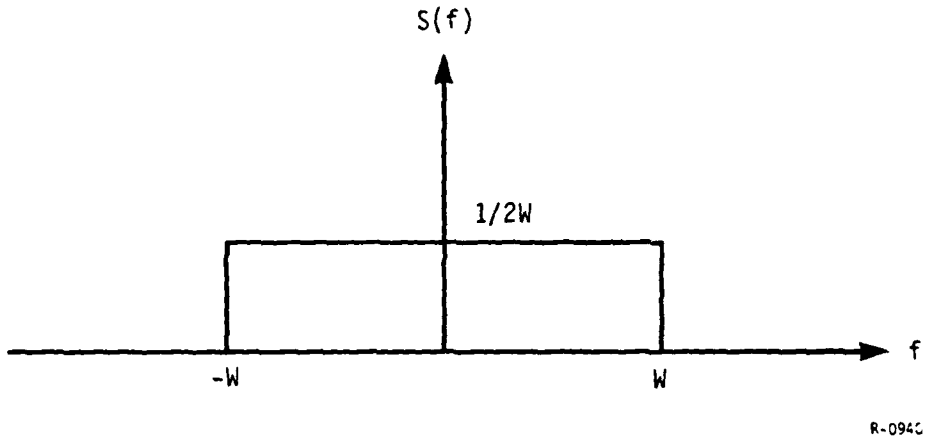

Figure 3-1. Signal Spectrum.

To determine the performance of the DLRT we first determine the form of the local likelihood ratios and then calculate their joint probability distribution function (conditioned on each hypothesis).

3.1.1 Local Log-Likelihood Ratio

If we expand the received signal via a K-L series,

$$y_i(t) = \lim_{K\to\infty} \sum_{k=1}^{K} y_i^k \phi^k(t) \qquad 0<t<T \tag{3-5}$$

where the $\phi^k(t)$ satisfy

$$\lambda^k \phi^k(t) = \int_0^T K^s(t,u)\phi^k(u)du \qquad 0<t<T \tag{3-6}$$

it is easy to show [2] that the log-likelihood ratio based on the first K coefficients at the i-th sensor is given by

$$\hat{\ell}_i^K = \frac{1}{2}\sum_{k=1}^{K}\left(\frac{c_i^2\lambda^k}{1+c_i^2\lambda^k}\right)(y_i^k)^2 - \frac{1}{2}\sum_{k=1}^{K}\ln(1+c_i^2\lambda^k) \quad . \tag{3-7}$$

Since the thresholds T_i are constants in Eq. 3-3 we shall henceforth work with only the data dependent portion of $\hat{\ell}_i^K$, which we define as

$$\ell_i^K \triangleq \frac{1}{2}\sum_{k=1}^{K}\left(\frac{c_i^2\lambda^k}{1+c_i^2\lambda^k}\right)(y_i^k)^2 \quad . \tag{3-8}$$

In the limit as $K\to\infty$ we obtain [2]

$$\ell_i = \frac{1}{2}\int_0^T\int_0^T y_i(t)h_i(t,u)y_i(u)dudt \tag{3-9}$$

where $h_i(t,u)$ satisfies

$$h_i(t,u) + \int_0^T c_i^2 h_i(t,z)K^s(z,u)dz = c_i^2K^s(t,u) \quad , \quad 0<t,u<T \quad . \tag{3-10}$$

3.1.2 Joint Probability Distribution Function

The derivation of expressions for the joint probability distribution functions of ℓ_1 and ℓ_2 conditioned on H^0 and H^1 is complex and thus the details are relegated to Appendix A. Here we briefly sketch the derivation.

We first compute the joint moment generating function of the ℓ_i^K. This is relatively straightforward as the y_i^k are (for each i) conditionally independent. We then take the limit as $K\to\infty$. If we define

$$\mu_i(r,s) \triangleq E\{e^{-r\ell_1 - s\ell_2} | H^i\} \tag{3-11}$$

then we have

$$\ln \mu_0(r,s) = \frac{1}{2} \sum_{k=1}^{\infty} \ln \left[\left(\frac{1+c_1^2\lambda^k}{1+(1+r)c_1^2\lambda^k} \right) \left(\frac{1+c_2^2\lambda^k}{1+(1+s)c_2^2\lambda^k} \right) \right] \tag{3-12}$$

$$\ln \mu_1(r,s) = \frac{1}{2} \sum_{k=1}^{\infty} \ln \left[\frac{(1-\rho_k^2)}{(1+r(1-\rho_k^2)c_1^2\lambda^k)(1+s(1-\rho_k^2)c_2^2\lambda^k)-\rho_k^2} \right] \tag{3-13}$$

where

$$\rho_k \triangleq \frac{c_1 c_2 \lambda^k}{\sqrt{(1+c_1^2\lambda k)(1+c_2^2\lambda k)}} \; . \tag{3-14}$$

If the observation time is long in comparison to the signal time constants (i.e., if WT is large enough) we have [1] that the infinite sums in Eqs. 3-12 and 3-13 can be replaced with integrals in which the eigenvalue magnitude appearing in Eqs. 3-12 and 3-13 is replaced by the signal's spectral height. This yields (for the ideal bandlimited signal) a trivial integral and we have

$$\mu_0(r,s) \cong \left[\frac{1+c_1^2/2W}{1+(1+r)c_1^2/2W}\right]^{WT}\left[\frac{1+c_2^2/2W}{1+(1+s)c_2^2/2W}\right]^{WT} \tag{3-15}$$

$$\mu_1(r,s) \cong \left[\frac{1-\rho^2}{(1+r(1-\rho^2)c_1^2/2W)(1+s(1-\rho^2)c_2^2/2W)-\rho^2}\right]^{WT} \tag{3-16}$$

where

$$\rho = \frac{c_1c_2/2W}{\sqrt{(1+c_1^2/2W)(1+c_2^2/2W)}} \quad . \tag{3-17}$$

We can then invert each moment generating function to obtain

$$p(\ell_1,\ell_2|H^0) = \left[\frac{1+\gamma_1}{\gamma_1}\;\frac{1+\gamma_2}{\gamma_2}\right]^{\Delta/2}\frac{(\ell_1\ell_2)^{\Delta/2-1}}{(\Gamma(\Delta/2))^2}\,e^{-\left(\frac{1+\gamma_1}{\gamma_1}\right)\ell_1-\left(\frac{1+\gamma_2}{\gamma_2}\right)\ell_2} \tag{3-18}$$

$p(\ell_1,\ell_2|H^1)$

$$= \frac{\left[(1-\rho^2)\gamma_1\gamma_2\right]^{-\Delta/2}}{\Gamma(\Delta/2)} e^{-\frac{\ell_1}{(1-\rho^2)\gamma_1} - \frac{\ell_2}{(1-\rho^2)\gamma_2}} \left[\frac{\sqrt{\ell_1\ell_2}(1-\rho^2)\gamma_1\gamma_2}{\rho}\right]^{\Delta/2-1}$$

$$\cdot\ I_{\Delta/2-1}\left(2\left[\frac{\rho}{(1-\rho^2)\gamma_1\gamma_2}\right]\sqrt{\ell_1\ell_2}\right) \tag{3-19}$$

where we have defined $\Delta=2WT$ as the observation time-signal bandwidth product, $E_i=c_i^2T$ as the signal energy received by the i-th sensor and $\gamma_i=E_i/\Delta$ as the signal-to-noise ratio in the signal bandwidth (recall the noise had unit spectral height) and where $I_\nu(\cdot)$ is the modified Bessel function of order ν.

Surprisingly, both Eqs. 3-18 and 3-19 can be integrated analytically when $\Delta/2-1$ is a nonnegative integer. We thus obtain under this assumption the joint distribution functions:

$$\Pr\left\{\ell_1<T_1,\ell_2<T_2|H^0\right\} = \left(1-e^{-\delta_1}\sum_{r=0}^{\Delta/2-1}\delta_1^r/r!\right)\left(1-e^{-\delta_2}\sum_{r=0}^{\Delta/2-1}\delta_2^r/r!\right) \tag{3-20}$$

where

$$\delta_i = \frac{1+c_i^2}{c_i^2}\,T_i \tag{3-21}$$

and

$\Pr\{\ell_1<T_1,\ell_2<T_2|H^1\} =$

$$(1-\rho^2)^{\Delta/2}\sum_{k=0}^{\infty}\frac{k+\Delta/2-1!}{k!\Delta/2-1!}\rho^{2k}\left[1-\sum_{r=0}^{\Delta/2-1}\beta_1^r/r!\right]\left[1-\sum_{r=0}^{\Delta/2-1}\beta_2^r/r!\right]e^{-(\beta_1+\beta_2)} \tag{3-22}$$

where

$$\beta_i = \frac{T_i}{(1-\rho^2)c_i^2} \quad . \tag{3-23}$$

Via Eqs. 3-20 and 3-22 we can evaluate the performance of the DLRT for any pair of thresholds. It is thus straightforward to determine via numerical techniques the optimal thresholds for any given problem. Results are given in the next section.

3.2 NUMERICAL EXAMPLES

Here we will present numerical results for the distributed detection of an ideal bandlimited signal in white Gaussian noise. Both the general Bayesian case and the case of surveillance system design will be considered.

3.2.1 Bayesian Formulation

Using the cost function defined in subsection 2.3.1 we obtain the results depicted in Figs. 3-2 and 3-3. In Fig. 3-2 we have plotted the expected cost of the optimal DLRT and that of the locally optimal test* (LOT) as functions

*For the cases considered here the sensors are identical and we define the LOT as the DLRT satisfying the additional constraint that $T_1=T_2$.

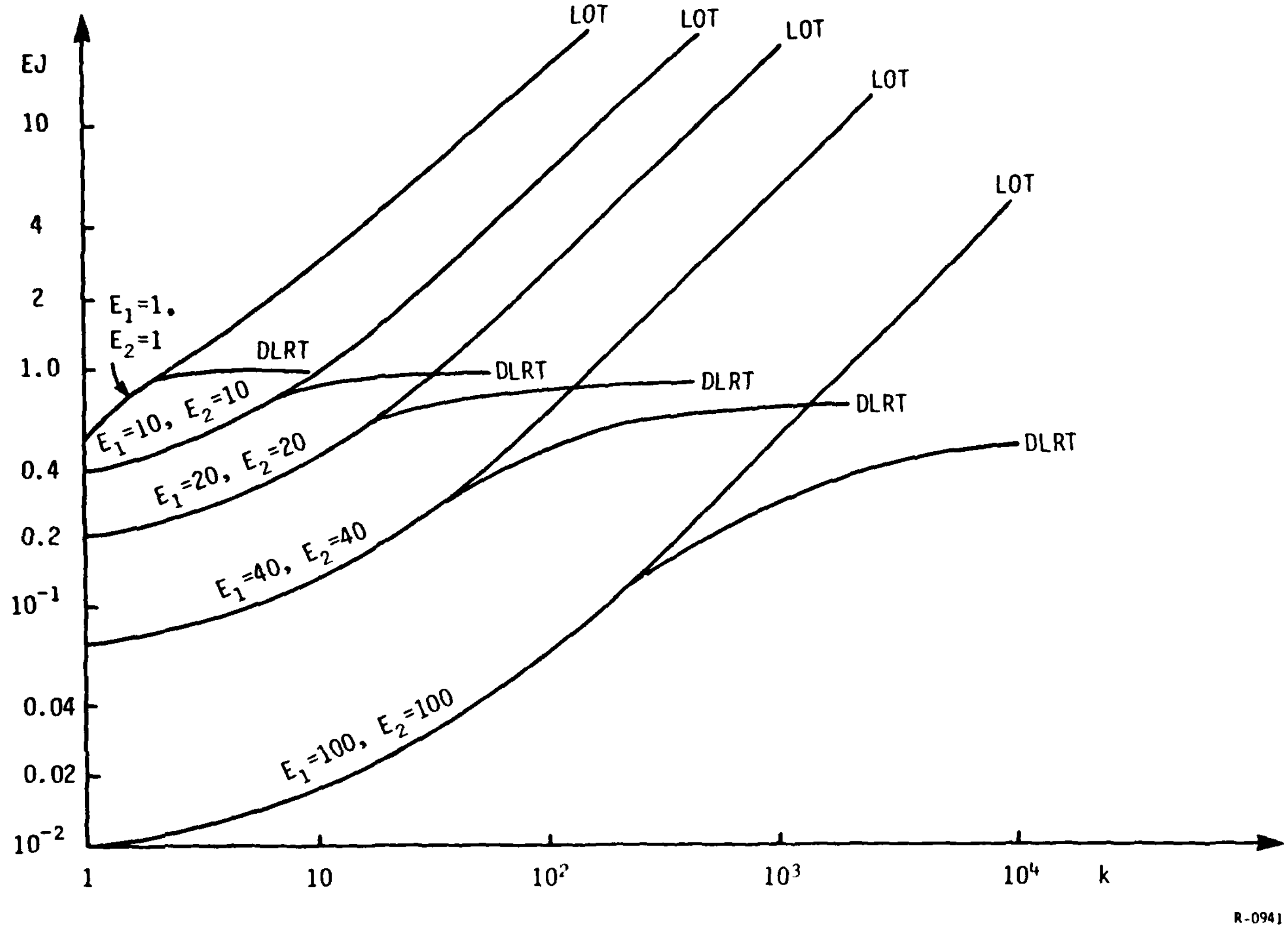

Figure 3-2. Bayes Cost Versus Double Error Cost, Time-Bandwidth Product Constant ($\Delta=10$), Signal Energy (E_1) Varying.

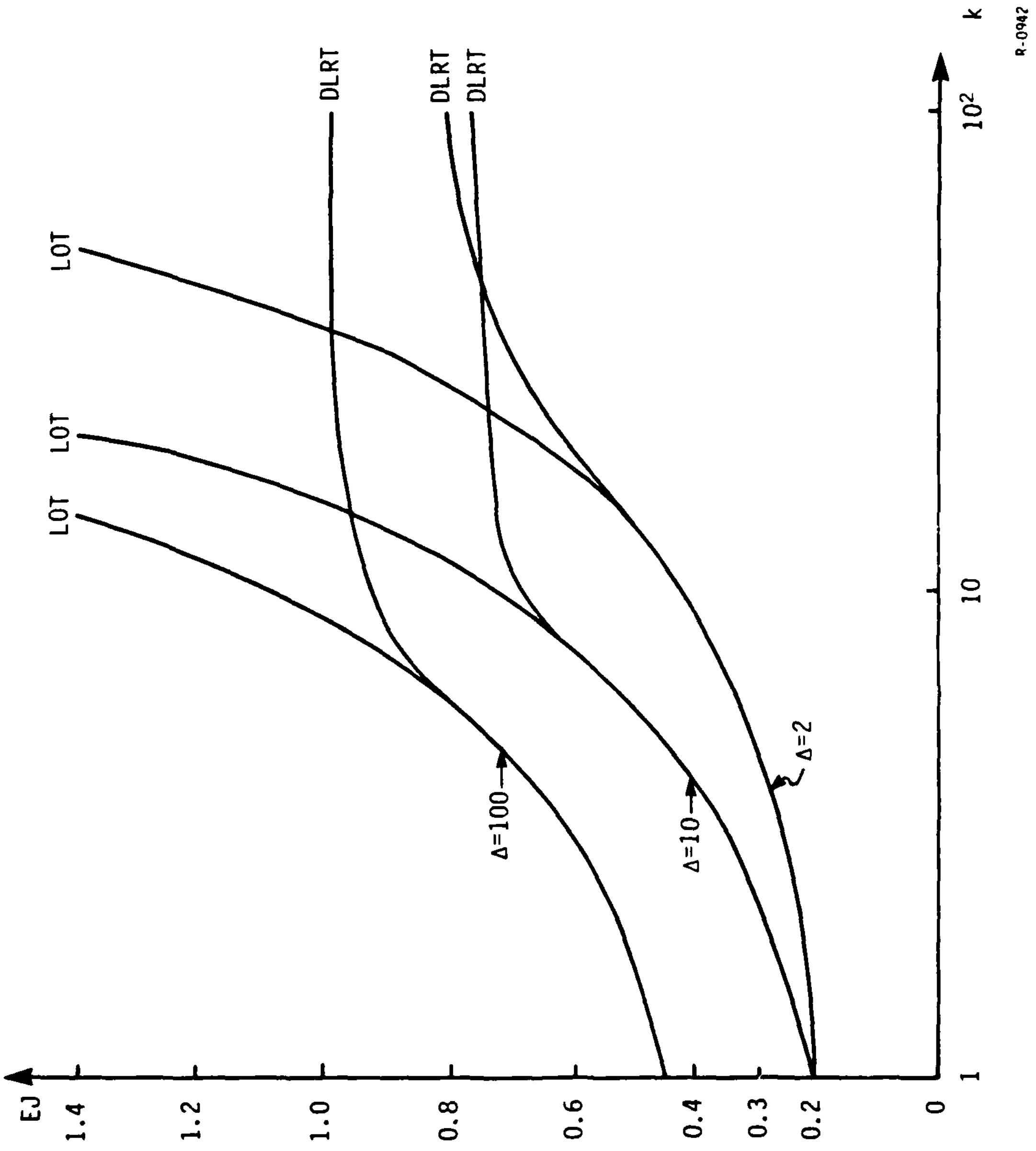

Figure 3-3. Bayes Cost Versus Double Error Cost, Signal Energy Constant (E_I=20), Time-Bandwidth Product (Δ) Varying.

of the double error cost k. The different curves correspond to different values of signal energy (E_i=1, 10, 20, 40, 100), where the time-bandwidth product has been held constant (Δ=10).

We note that identical results are obtained for $k \leq 2$, but that the costs associated with the LOT become affine for large enough k. This occurs because when $T_1=T_2$ the probability of a double error cannot be reduced to zero. For large enough k, the cost of double errors dominates the growth of the expected value of the cost EJ and the affine curves result. The cost for the optimal DLRT of course never takes on a value greater than unity (since taking $u_1=1$ and $u_2=0$ gives $J \leq 1$). As the signal energy is increased the performance of the DLRT improves and the point at which the LOT is not longer globally optimal is also increased. This latter effect occurs because, as the signal energy increases, the probability of a double error decreases rapidly. This means that the cost is dominated by the cost of single errors until k becomes very large.

In Fig. 3-3 we again plot the performance of the LOT and the DLRT versus k, but now signal energy is held constant, E_i=20, and the time-bandwidth product is allowed to vary (Δ=2, 10, 100). Here we see the effect of frequency diversity - for small k the LOT is optimal and the DLRT performance for Δ=10 is better than either Δ=2 or Δ=100 (as also in Figs. 3-4 and 3-5). As k increases, the effect of double errors increases and DLRT performance is dominated by the need to minimize double errors. For large k the DLRT skews the thresholds so that one detector is likely to have u=0 and the other have u=1. This strategy is so effective that, for the cases of Fig. 3-3 with k>20, the cost of double errors is less than 5 percent of the total cost. The structure of the Bayes cost EJ for large k is thus determined by the probability of single errors when the thresholds are skewed to avoid double errors.

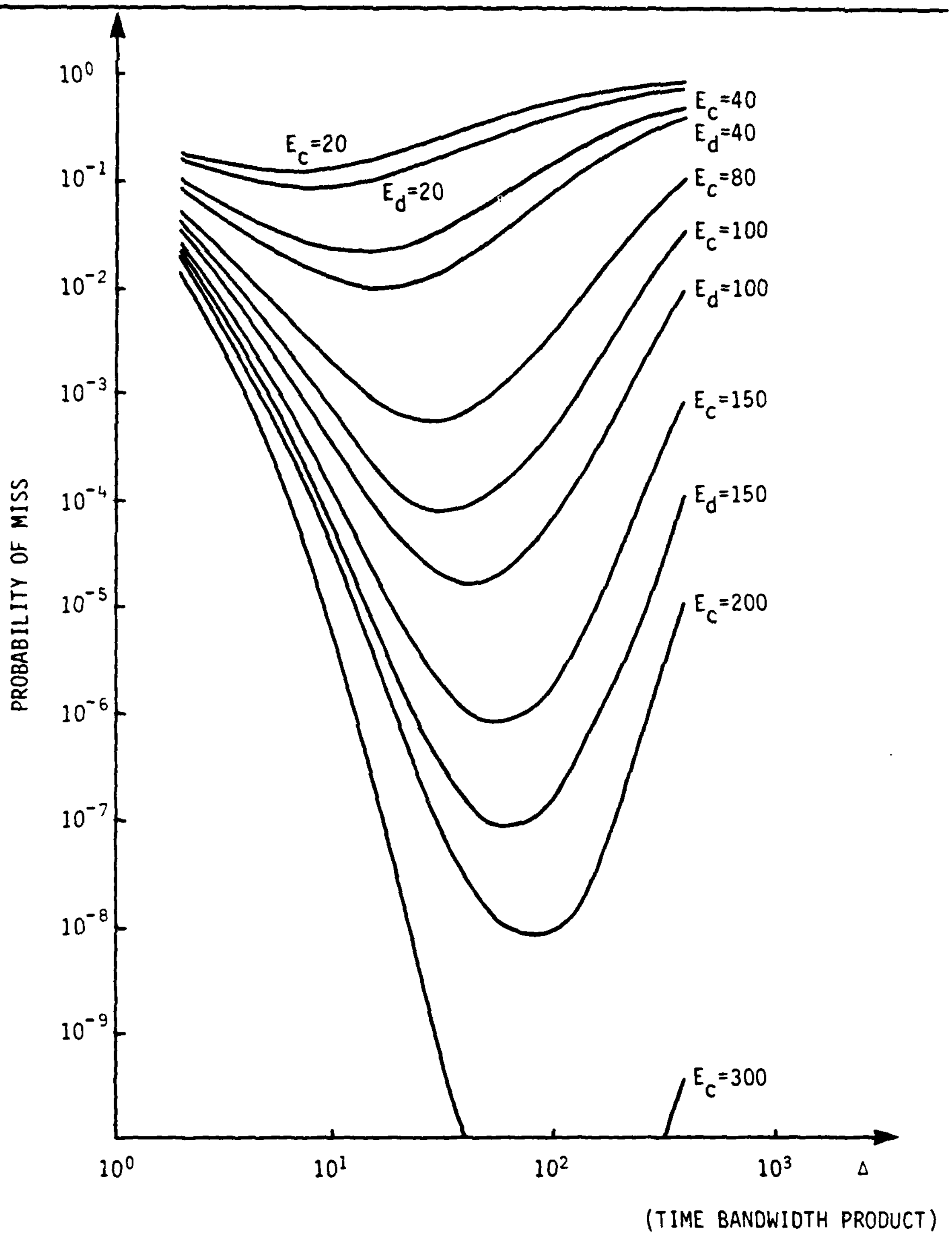

Figure 3-4. Plot of P_M for DLRT When P_F=0.1.

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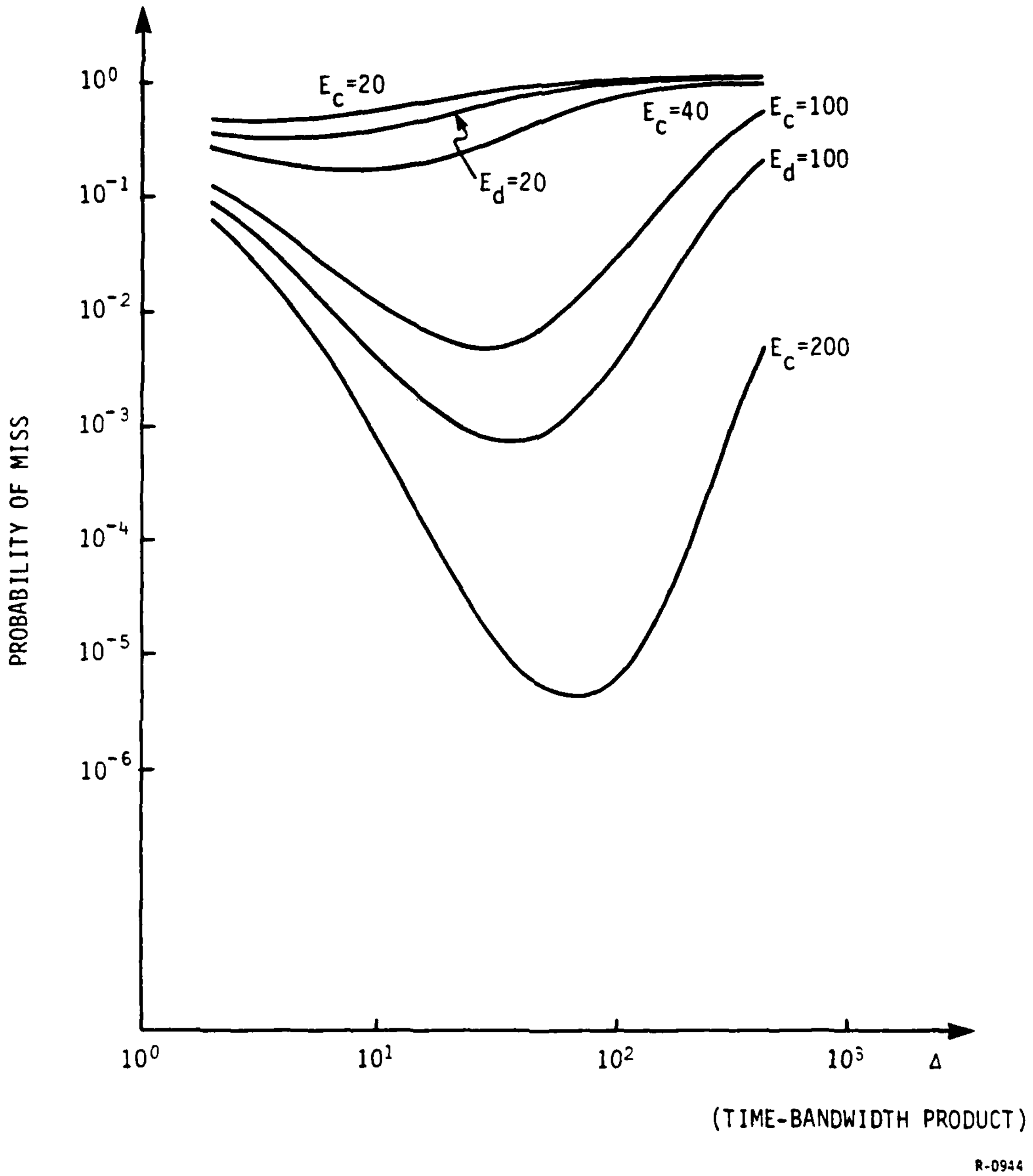

Figure 3-5. Plot of P_M for DLRT When P_F=0.001.

3.2.2 Surveillance System Design

Now we present results for the case described in subsection 2.3.2, i.e., the case in which a global decision is made at a fusion center by "adding" together the local decisions made from the observations of the individual sensors.

In Figs. 3-4 and 3-5 we plot the probability of a miss (i.e., that H^0 is decided when H^1 is true) as a function of the time-bandwidth product Δ=2WT, for fixed probability of false alarm (i.e., that H^1 is decided when H^0 is true). In Fig. 3-4 we have set P_F=0.1 and in Fig. 3-5 we have set P_F=0.001. The various curves represent the performance of the DLRTs and centralized LRTs for different signal energies.

The curves labeled E_d plot the DLRT performance for two sensors each of which, under H^1, observes a signal of energy E_d. The curves labeled E_c plot the performance of the optimal centralized likelihood ratio test where, under H^1, the total signal energy received is E_c.* In both figures we see the anticipated ranking of performance curves: $E_c < E_d < 2E_c$. This occurs since the DLRT for the two sensor case must perform at least as well as the one sensor centralized LRT and cannot perform as well as the two sensor centralized LRT.

In both these figures we see that the performance initially improves as Δ increases and then degrades. This is basically due to the effect of frequency diversity. We know [2] that most of the signal energy is associated with the first $2\Delta+1$ eigenfunctions $\phi^k(t)$. Thus as Δ increases the number of significant independent "observations" of the signal increases, however, the

*For this problem the centralized performance of one sensor receiving E_c is exactly equivalent to that of two sensors each receiving $E_c/2$.

signal-to-noise ratio (SNR) for each "observation" decreases. Thus these figures plot the tradeoff of number of observations versus SNR per observation.

We note from Figs. 3-4 and 3-5 that the performance of the optimal DLRT seems to be more similar to that of the one sensor centralized LRT than to that of the two sensor LRT. In Fig. 3-6 we consider the relative performance of these systems more closely. We compare the centralized and decentralized systems by determining the amount of signal energy required by a centralized system to perform as well as a decentralized system. The ratio E_c/E_d is largest when the decentralized system performs well and is smallest when it performs poorly. We take as baseline the performance of the optimal DLRT with $E_d=100$ and $P_F=0.001$ and determine the E_c required to obtain the equivalent P_M. Note that for $\Delta=2$, $E_c/E_d \cong 1.40$ and thus the two-sensor DLRT performs as well as 1.40 centralized sensors. As Δ increases the DLRT performance degrades until at $\Delta=100$ we have $E_c/E_d \cong 1.25$.

The performance of the DLRT begins to improve as 2WT increases above 100. The asymptotic performance has not been determined as optimal performance has not been determined analytically and, for 2WT>100, numerical difficulties intrude into the computation of P_M and P_F.

In Fig. 3-7 we plot the performance of DLRTs for different combinations of sensors with $P_F=0.001$. If we let E_d^i denote the signal energy received by the i-th sensor than for all curves we have $E_d^1+E_d^2=200$. The different curves correspond to various ratios E_d^1/E_d^2. We note that the performance increases as the sensors become more asymmetric. This occurs since we are effectively moving toward a centralized solution (note that at $E_d^1=0$, $E_d^2=200$ the

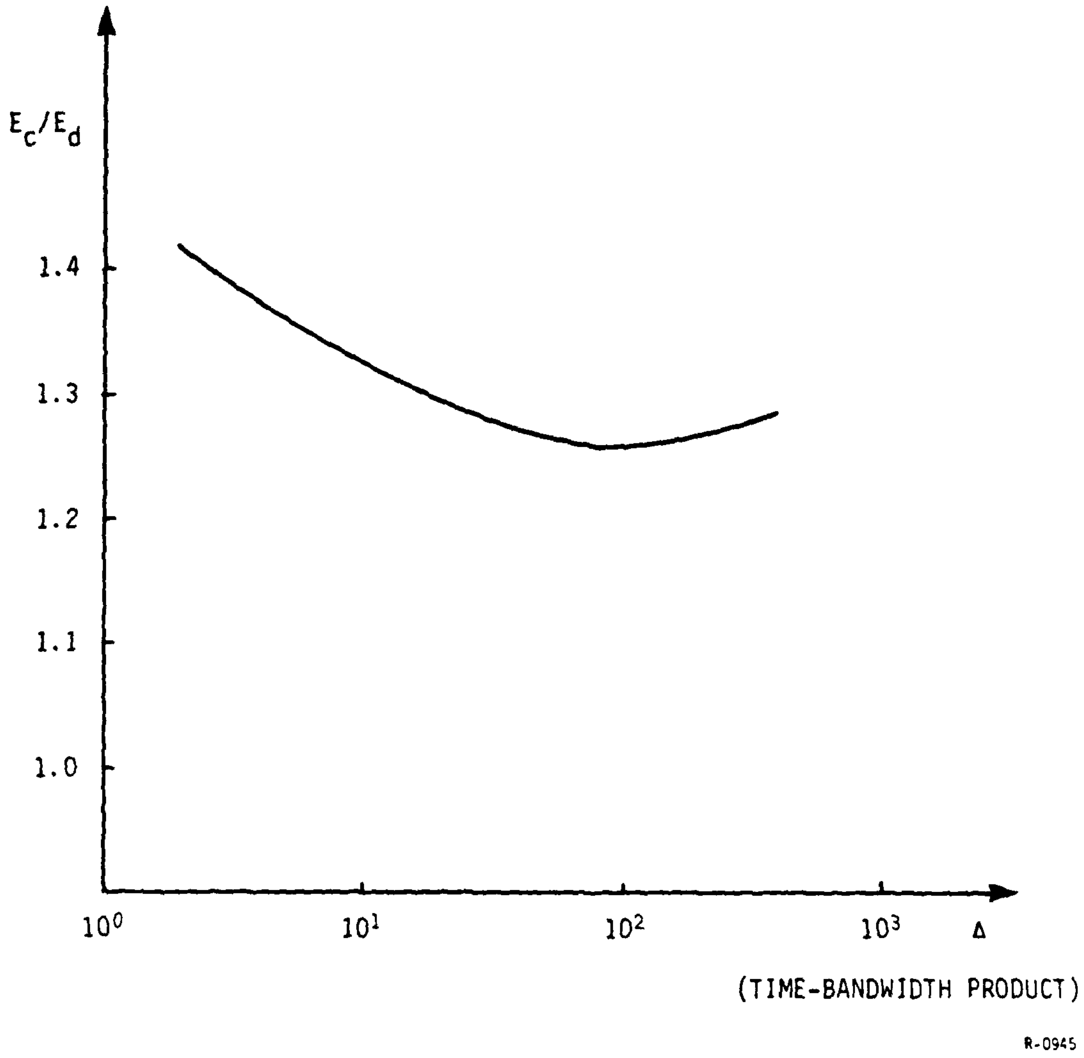

Figure 3-6. Ratio of Energy Required by a Single Sensor to Equal the Performance of an Optimal DLRT with E_d=100 and P_F=0.001.

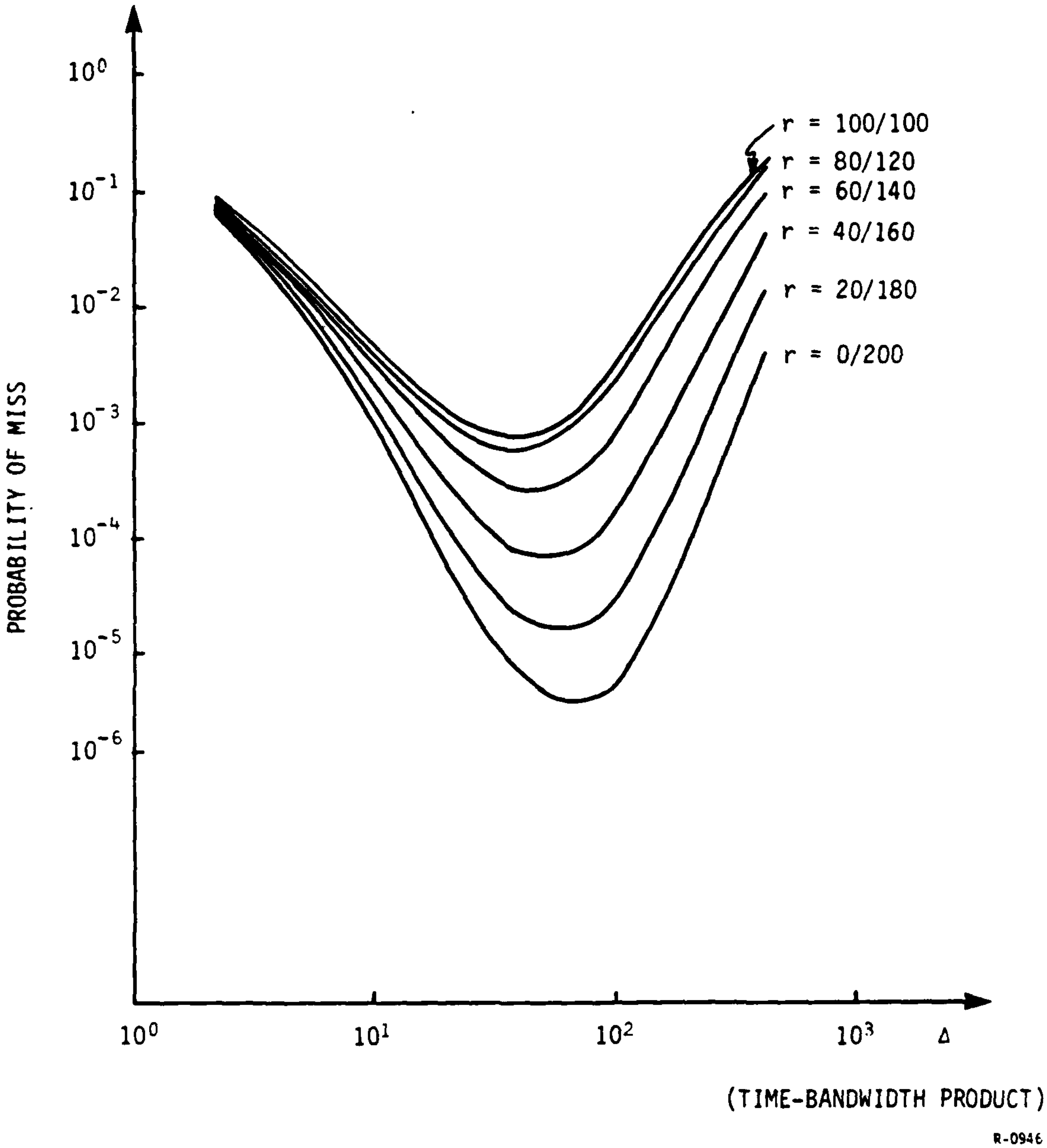

Figure 3-7. Plot of P_M for DLRT with P_F=0.001, Various Sensor Combinations.

performance is identical to that of E_c=200). When E_d^1=0 and E_d^2=200 the first threshold is zero so that u_1=1 always and the second threshold is identical to that associated with E_c=200.

SECTION 4

SUMMARY, CONCLUSIONS, AND SUGGESTED RESEARCH

In this paper we have considered the problem of distributed detection with limited communication. We have obtained optimal detection laws for the case of known signals in noise uncorrelated between sensors and for the case of linearly dependent known signals in white noise correlated between sensors. For both of these cases the optimal distributed detection law consists of forming local likelihood ratios and testing these ratios against thresholds to determine a decision. For uncorrelated noise a single threshold is optimal. This may well be the case for correlated noise as well, but we were unable to prove or disprove this conjecture.

For a specific example, with noise correlated between sensors, we found that the distributed detection law performs worse as the correlation increases, while the centralized law may either perform better or worse. Thus an analysis is necessary in each case to determine whether the performance penalty for implementing a distributed detection law is worth the payoff in terms of reduced communications requirements.

We then investigated the problem of distributed detection of an unknown signal in noise. This problem was found to be considerably more difficult than the case of a known signal since the detection law depends upon the entire received signal - a scalar sufficient statistic does not exist in general. As the optimal distributed detection law could not be determined we

investigated laws employing local likelihood ratio tests. Even for this case results are difficult to obtain since the local log-likelihood ratios are not Gaussian. However, for an important special case (long observation time, ideal bandlimited signals, white sensor noise) we were able to obtain analytical results. For this case we found that two identical distributed sensors generally perform as well as 1.25 to 1.40 centralized sensors.

More general problems of unknown signals in noise do not appear solvable by using the methods of this report since the exact probability distribution function is generally difficult to determine. The approach used in centralized problems [1],[2] is to generate approximations based on Chernov bounds and Gaussian approximations or Edgeworth expansions. We considered several such approximations for the distributed problem - in all cases the approximations were too inaccurate to be useful. A further drawback to those approximations is that the moment generating function $\mu_1(r,s)$ is required and, as Appendix A makes clear, these quantities are difficult to compute for all but the ideal bandlimited case.

Work in distributed detection is in its infancy, and many of the issues that arise in the centralized theory can also be formulated in the distributed setting. Moreover, the increasing tendency to net sensors together in military surveillance systems makes the distributed detection problem formulation of potential practical significance. However, as the results of this report make clear, the distributed version of a given detection problem is often significantly more difficult to solve than its centralized counter part.

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APPENDIX A

DERIVATION OF LRT PROBABILITY DISTRIBUTIONS

In this appendix we derive the joint probability distribution of the log-likelihood ratios considered in Eq. 3-8. Recall that the log-likelihood ratios ℓ_i are defined as

$$\ell_i = \lim_{K\to\infty} \ell_i^K \tag{A-1}$$

where

$$\ell_i^K \triangleq \frac{1}{2} \sum_{k=1}^{K} \left(\frac{c_i^2 \lambda^k}{1+c_i^2 \lambda^k} \right) (y_i^k)^2 \tag{A-2}$$

and the λ^k are the eigenvalues of $K^s(t,u)$ where

$$K^s(t,u) = E\{s(t)s(u)\} \quad . \tag{A-3}$$

The y_i^k are the Karhunen-Loeve coefficients associated with the k-th eigenfunction of K^s. It is straightforward to verify that

$$E\{y_i^k | H^j\} = 0 \quad , \quad i=1,2, j=0,1 \tag{A-4}$$

$$E\{y_i^j y_i^k | H^0\} = \delta(j-k) \tag{A-5}$$

$$E\{y_1^j y_2^k | H^0\} = 0 \tag{A-6}$$

$$E\{y_i^j y_i^k | H^1\} = (c_i^2 \lambda^j + 1)\delta(j-k) \tag{A-7}$$

$$E\{y_1^j y_2^k | H^1\} = c_1 c_2 \lambda^j \delta(j-k) \quad . \tag{A-8}$$

Thus, recalling that

$$\mu_i^K(r,s) \triangleq E\{e^{-r\ell_1^K - s\ell_2^K} | H^i\} \quad , \quad i=0,1 \quad , \tag{A-9}$$

we have

$$\mu_0^K(r,s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-r\frac{1}{2}\sum_{k=1}^{K}\left[\frac{c_1^2\lambda^k}{1+c_1^2\lambda^k}\right](y_1^k)^2 - s\frac{1}{2}\sum_{k=1}^{K}\left[\frac{c_2^2\lambda^k}{1+c_2^2\lambda^k}\right](y_2^k)^2}$$

$$\frac{1}{(2\pi)^K} e^{-\frac{1}{2}\sum_{k=1}^{K}(y_1^k)^2 - \frac{1}{2}\sum_{k=1}^{K}(y_2^k)^2} dy_1^1,\ldots,dy_2^K$$

$$= \prod_{k=1}^{K} \int_{-\infty}^{\infty} \frac{\sqrt{\dfrac{1+c_1^2\lambda^k}{1+(1+r)c_1^2\lambda^k}}}{\sqrt{2\pi\dfrac{1+c_1^2\lambda^k}{1+(1+r)c_1^2\lambda^k}}} e^{-\frac{1}{2}\left[\frac{1+c_1^2\lambda^k}{1+(1+s)c_1^2\lambda^k}\right]^{-1}(y_1^k)^2} dy_1^k \quad . \tag{A-10}$$

(continued)

$$\int_{-\infty}^{\infty} \frac{\sqrt{\frac{1+c_2^2\lambda^k}{1+(1+s)c_2^2\lambda^k}}}{\sqrt{2\Pi \frac{1+c_2^2\lambda^k}{1+(1+s)c_2^2\lambda^k}}} e^{-\frac{1}{2}\left[\frac{1+c_2^2\lambda^k}{1+(1+s)c_2^2\lambda^k}\right]^{-1} (y_2^k)^2} dy_2^k$$

$$= \prod_{k=1}^{K} \sqrt{\frac{1+c_1^2\lambda^k}{1+(1+r)c_1^2\lambda^k}} \sqrt{\frac{1+c_2^2\lambda^k}{1+(1+s)c_2^2\lambda^k}} , \qquad \text{(A-10)}$$

and similarly

$$\mu_1^K(r,s) = \prod_{k=1}^{K} \sqrt{\frac{1-\rho_k^2}{(1+r(1-\rho_k^2)c_1^2\lambda^k)(1+s(1-\rho_k^2)c_2^2\lambda^k)-\rho_k^2}} \qquad \text{(A-11)}$$

where

$$\rho_k \triangleq \frac{c_1 c_2 \lambda^k}{\sqrt{(1+c_1^2\lambda k)(1+c_2^2\lambda k)}} . \qquad \text{(A-12)}$$

The μ_i^K are not in a form which allows inversion and thus we simplify Eqs. A-10 and A-11. We do this by noting that [1], if

$$g_\lambda \triangleq \sum_{i=1}^{\infty} g(\lambda_i) \qquad \text{(A-13)}$$

where λ_i are the eigenvalues associated with K(t,u), then for long enough observation time T

$$g_\lambda \cong T \int_{-\infty}^{\infty} g(S(f))df \tag{A-14}$$

where S(f) is the spectrum of K(t,u). The μ_i can be written as in Eq. A-13 by using Eq. A-14 if we first take the logarithm and the limit as $K\to\infty$. Thus

$$\ell n\ \mu_0(r,s) = \frac{1}{2}\sum_{k=1}^{\infty} \ell n\left(\frac{1+c_1^2\lambda_k}{1+(1+r)c_1^2\lambda_k}\quad\frac{1+c_2^2\lambda_k}{1+(1+s)c_2^2\lambda_k}\right)$$

$$\cong \frac{T}{2}\int_{-\infty}^{\infty} \ell n\left(\frac{1+c_1^2S(f)}{1+(1+r)c_1^2S(f)}\quad\frac{1+c_2^2S(f)}{1+(1+s)c_2^2S(f)}\right)df \tag{A-15}$$

and

$$\ell n\ \mu_1(r,s) = \frac{1}{2}\sum_{k=1}^{\infty} \ell n\left(\frac{1-\rho_k^2}{(1+r(1-\rho_k^2)c_1^2\lambda_k)(1+s(1-\rho_k^2)c_2^2\lambda_k)-\rho_k^2}\right)$$

$$\cong \frac{T}{2}\int_{-\infty}^{\infty} \ell n\left(\frac{1-\rho_k^2}{(1+r(1-\rho^2(f))c_1^2S(f))(1+s(1-\rho^2(f))c_2^2S(f)-\rho^2(f)}\right)df \tag{A-16}$$

where

$$\rho(f) \triangleq \frac{c_1 c_2 S(f)}{\sqrt{(1+c_1^2 S(f))(1+c_2^2 S(f))}} . \tag{A-17}$$

The integrals in Eqs. A-15 and A-16 are readily evaluated for the ideal bandlimited spectrum, yielding (after exponentiation)

$$\mu_0(r,s) = \left[\frac{1+c_1^2/2W}{1+(1+r)c_1^2/2W}\right]^{WT} \left[\frac{1+c_2^2/2W}{1+(1+s)c_2^2/2W}\right]^{WT} \tag{A-18}$$

and

$$\mu_1(r,s) = \left[\frac{1-\rho^2}{(1+r(1-\rho^2)c_1^2/2W)(1+s(1-\rho^2)c_2^2/2W)-\rho^2}\right]^{WT} \tag{A-19}$$

where

$$\rho = \frac{\sqrt{c_1^2/2W \quad c_2^2/2W}}{\sqrt{(1+c_1^2/2W)(1+c_2^2/2W)}} . \tag{A-20}$$

We define $E_i = c_i^2 T$ as the signal energy received by the i-th sensor, $\Delta = 2WT$ as the observation time-signal bandwidth product, and $\gamma_i = c_i^2 T/2WT$ as the signal-to-noise ratio in the signal bandwidth. With these definitions we have obtained Eqs. 3-15 and 3-16 of Section 3:

$$\mu_0(r,s) = \left[\frac{1+\gamma_1}{1+(1+r)\gamma_1}\right]^{\Delta/2}\left[\frac{1+\gamma_2}{1+(1+s)\gamma_2}\right]^{\Delta/2} \tag{A-21}$$

$$\mu_1(r,s) = \left[\frac{1-\rho^2}{(1+r(1-\rho^2)\gamma_1)(1+s(1-\rho^2)\gamma_2)-\rho^2}\right]^{\Delta/2} . \tag{A-22}$$

Using a standard set of Laplace transforms [5] we can invert Eq. A-21 to obtain

$$p(\ell_1,\ell_2|H^0) = \left[\frac{1+\gamma_1}{\gamma_1}\quad\frac{1+\gamma_2}{\gamma_2}\right]^{\Delta/2} \frac{(\ell_1\ell_2)^{\Delta/2-1}\, e^{-\left(\frac{1+\gamma_1}{\gamma_1}\right)\ell_1-\left(\frac{1+\gamma_2}{\gamma_2}\right)\ell_2}}{(\Gamma(\Delta/2))^2} . \tag{A-23}$$

Inverting Eq. A-22 on r first yields

$$p(\ell_1,s|H^1)$$

$$= \left[\frac{1}{\gamma_1(1+s(1-\rho^2)\gamma_2)}\right]^{\Delta/2} \frac{(\ell_1)^{\Delta/2-1}\, e^{-\frac{\ell_1}{(1-\rho^2)\gamma_1}}}{\Gamma(\Delta/2)}\, e^{\frac{\rho^2\ell_1}{(1-\rho^2)\gamma_1(1+s(1-\rho^2)\gamma_2)}}$$

$$= \frac{(\ell_1)^{\Delta/2-1}}{\Gamma(\Delta/2)}\, e^{-\frac{\ell_1}{(1-\rho^2)\gamma_1}}\left[\frac{1}{\gamma_1\gamma_2(1-\rho^2)}\right]^{\Delta/2} \tag{A-24}$$

(continued)

$$\cdot \left[\frac{1}{s+1/(\gamma_2(1-\rho^2))}\right]^{\Delta/2} e^{\frac{\rho^2 \ell_1}{(1-\rho^2)\gamma_1\gamma_2} \frac{1}{s+1/((1-\rho^2)\gamma_2)}} \tag{A-24}$$

which, using the time shift property of Laplace transforms, yields

$$p(\ell_1,\ell_2|H^1) = \frac{(\ell_1)^{\Delta/2-1}}{\Gamma(\Delta/2)} e^{-\frac{\ell_1}{(1-\rho^2)\gamma_1} - \frac{\ell_2}{(1-\rho^2)\gamma_2}} \left[\frac{1}{(1-\rho^2)\gamma_1\gamma_2}\right]^{\Delta/2}$$

$$\cdot\ \ell^{-1}\left\{\frac{e^{\frac{\rho^2\ell_1}{(1-\rho^2)\gamma_1\gamma_2}\frac{1}{s}}}{s^{\Delta/2}}\right\} . \tag{A-25}$$

The inverse transform is standard and yields

$$p(\ell_1,\ell_2|H^1) = \left[\frac{1}{(1-\rho^2)\gamma_1\gamma_2}\right]^{\Delta/2} \frac{e^{-\frac{1}{1-\rho^2}\left(\frac{\ell_1}{\gamma_1}+\frac{\ell_2}{\gamma_2}\right)}}{\Gamma(\Delta/2)} \left[\frac{\sqrt{\ell_1\ell_2(1-\rho^2)\gamma_1\gamma_2}}{\rho}\right]^{\Delta/2-1} \tag{A-26}$$

$$\cdot\ I_{\Delta/2-1}\left(2\left(\frac{\rho}{(1-\rho^2)\gamma_1\gamma_2}\right)\sqrt{\ell_1\ell_2}\right)$$

where $I_\nu(\cdot)$ is the modified Bessel function of order ν.

To obtain the distribution functions we now integrate Eqs. A-23 and A-26. To perform these integrations we assume that $\Delta/2-1$ is a nonnegative integer. We obtain under H^0:

$$\Pr\{\ell_1<T_1,\ell_2<T_2|H^0\} = \int_0^{T_1}\int_0^{T_2} p(\ell_1,\ell_2|H^0)d\ell_1 d\ell_2 = \left[\frac{1+\gamma_1}{\gamma_1}\;\frac{1+\gamma_2}{\gamma_2}\right]^{\Delta/2}$$

$$\cdot \frac{1}{(\Gamma(\Delta/2))^2}\int_0^{T_1} \ell_1^{\Delta/2-1} e^{-\left(\frac{1+\gamma_1}{\gamma_1}\right)\ell_1} d\ell_1 \int_0^{T_2} \ell_2^{\Delta/2-1} e^{-\left(\frac{1+\gamma_2}{\gamma_2}\right)\ell_2} d\ell_2$$

$$= \left[\frac{1+\gamma_1}{\gamma_1}\;\frac{1+\gamma_2}{\gamma_2}\right]^{\Delta/2} \frac{1}{(\Gamma(\Delta/2))^2}\, e^{-\left(\frac{1+\gamma_1}{\gamma_1}\right)\ell_1} \sum_{r=0}^{\Delta/2-1} \frac{r!\,\ell_1^{\Delta/2-1-r}}{(\Delta/2-1-r)!}\left(\frac{\gamma_1}{1+\gamma_1}\right)^{r+1}\Bigg|_0^{T_1}$$

$$\cdot\, e^{-\left(\frac{1+\gamma_2}{\gamma_2}\right)\ell_2} \sum_{r=0}^{\Delta/2-1} \frac{r!\,\ell_2^{\Delta/2-1-r}}{(\Delta/2-1-r)!}\left(\frac{\gamma_2}{1+\gamma_2}\right)^{r+1}\Bigg|_0^{T_2}$$

$$= e^{-(\delta_1+\delta_2)}\left(e^{\delta_1} - \sum_{r=0}^{\Delta/2-1}\delta_1^r/r!\right)\left(e^{\delta_2}\sum_{r=0}^{\Delta/2-1}\delta_2^r/r!\right) \tag{A-27}$$

where

$$\delta_i = \frac{1+\gamma_i}{\gamma_i}T_i \quad , \quad i=1,2 \quad . \tag{A-28}$$

To integrate $p(\ell_1,\ell_2|H^1)$ we expand $I_\nu(\cdot)$ in a series and integrate term by term. From [13] we have that

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{2k} / k!(\nu+k)! \tag{A-29}$$

and thus we obtain

$$p(\ell_1,\ell_2|H^1) = \frac{1}{(1-\rho^2)\gamma_1\gamma_2}$$

$$\cdot \left[\frac{1-\rho^2}{\rho^2}\right]^{\Delta/2-1} \frac{e^{-\frac{1}{1-\rho^2}\left(\frac{\ell_1}{\gamma_1}+\frac{\ell_2}{\gamma_2}\right)}}{\Gamma(\Delta/2)} \sum_{k=0}^{\infty} \frac{\left(\frac{\rho^2\ell_1\ell_2}{(1-\rho^2)^2\gamma_1\gamma_2}\right)^{k+\Delta/2-1}}{k!k+\Delta/2-1!} \cdot \tag{A-30}$$

Integration yields

$$\Pr\{\ell_1<T_1,\ell_2<T_2,|H^1\} = \int_0^{T_1}\int_0^{T_2} p(\ell_1,\ell_2|H^1)d\ell_1 d\ell_2$$

$$= \left(\frac{1-\rho^2}{\rho^2}\right)^{\Delta/2-1} \frac{1}{(1-\rho^2)\gamma_1\gamma_2}\,\frac{1}{\Gamma(\Delta/2)} \sum_{k=0}^{\infty} \frac{1}{k!\Delta/2-1+k!}$$

$$\cdot \left\{\int_0^{T_1} \left(\frac{\rho\ell_1}{(1-\rho^2)\gamma_1}\right)^{k+\Delta/2-1} e^{-\frac{\ell_1}{(1-\rho^2)\gamma_1}} d\ell_1\right\} \cdot$$

$$\cdot \left\{\int_0^{T_2} \left(\frac{\rho\ell_2}{(1-\rho^2)\gamma_2}\right)^{k+\Delta/2-1} e^{-\frac{\ell_2}{(1-\rho^2)\gamma_2}} d\ell_2\right\} \tag{A-31}$$

(continued)

$$= \left(\frac{1-\rho^2}{\rho^2}\right)^{\Delta/2-1} \frac{1}{(1-\rho^2)\gamma_1\gamma_2} \sum_{k=0}^{\infty} \frac{k+\Delta/2-1!}{k!\,\Delta/2-1!} \left(\frac{\rho^2}{(1-\rho^2)\gamma_1\gamma_2}\right)^{k+\Delta/2-1}$$

$$\cdot \left\{ e^{-\frac{\ell_1}{(1-\rho^2)\gamma_1}} \sum_{r=0}^{k+\Delta/2-1} \frac{(k+\Delta/2-1)!}{(k+\Delta/2-1-r)!} \frac{\left(\frac{\ell_1}{(1-\rho^2)\gamma_1}\right)^r}{r!} \Bigg|_0^{T_1} \right\}$$

$$\cdot \left\{ e^{-\frac{\ell_2}{(1-\rho^2)\gamma_2}} \sum_{r=0}^{k+\Delta/2-1} \frac{(k+\Delta/2-1)!}{(k+\Delta/2-1-r)!} \frac{\left(\frac{\ell_2}{(1-\rho^2)\gamma_2}\right)^r}{r!} \Bigg|_0^{T_2} \right\}$$

$$= (1-\rho^2)^{\Delta/2} \sum_{k=0}^{\infty} \frac{k+\Delta/2-1!}{k!\,\Delta/2-1!} \rho^{2k} \left\{ e^{-\beta_1} - e^{-\beta_1} \sum_{r=0}^{k+\Delta/2-1} \beta_1^r/r! \right\}$$

$$\cdot \left\{ e^{-\beta_2} - e^{-\beta_2} \sum_{r=0}^{k+\Delta/2-1} \beta_2^r/r! \right\} \qquad \text{(A-31)}$$

where

$$\beta_i = \frac{T_i}{(1-\rho^2)\gamma_i} \; . \qquad \text{(A-32)}$$

We thus have derived closed-form expressions for the joint probability distribution functions as desired.

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